21-301 Combinatorics
Homework 4
Due: Wednesday, October 6

1. Let $\mathcal{A}$ be a family of sub-sets of $[n]$. We say that $\mathcal{A}$ is *2-secure* if there do not exist $A, B, C, D \in \mathcal{A}$ such that $A \cap B \subseteq C \cup D$. Use the probabilistic method to show the existence of a 2-secure family of exponential size.
Solution: choose $A_1, A_2, \ldots, A_m$ uniformly at random. For a fixed i, j, k, ℓ, we have
$$\Pr(A_i \cap A_j) \subseteq (A_k \cup A_\ell)) = \left(\frac{15}{16}\right)^n.$$
It follows that there exists a 2-secure family of size m if $m^4 \left(\frac{15}{16}\right)^n < 1$.

2. Let $G = (V, E)$ be a graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose that for every v and $c \in S(v)$ there are at most d neighbors u of v such that c lies in $S(u)$. Use the local lemma to prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$. (By proper we mean that adjacent vertices get distinct colors.)
Solution: Assume that each list $S(v)$ is of size exactly $10d$. Randomly color each vertex v with a color c_v from its list $S(v)$. For each edge $e = \{v, w\}$ and color $c \in S(v) \cap S(w)$ we let $\mathcal{E}_{e,c}$ be the event that $c_v = c_w = c$. Thus $P(\mathcal{E}_{e.c}) = 1/(10d)^2$.

   Note that $\mathcal{E}_{\{v,w\},c}$ depends only on the colors assigned to v and w, and is thus independent of $\mathcal{E}_{\{v',w'\},c'}$ if $\{v', w'\} \cap \{v, w\} = \emptyset$. Hence $\mathcal{E}_{\{v,w\},c}$ only depends on other edges involving v or w. Now there are at most $10d \times d$ events $\mathcal{E}_{\{v,w'\},c'}$ where $c' \in S(v) \cap S(w')$. So the maximum degree in the dependency graph is at most $20d^2$. The result follows from $4 \times 20d^2 \times 1/(10d)^2 < 1$.

3. Show that if $8nk2^{1-k} < 1$ then one can 2-color the integers $1, 2, \ldots, n$ such that there is no mono-colored arithmetic progression of length k.
(An arithmetic progression of length k is a set $\{a, a + d, \ldots, a + (k - 1)d\}$.)

   Solution: Color the integers randomly. For an arithmetic progression $S = \{a, a + d, \ldots, a + (k - 1)d\}$ of length k, let $\mathcal{E}_S$ denote the event that S is mono-colored. Then $\Pr(\mathcal{E}_S) = 2^{-(k-1)}$.

   Now consider the dependency graph of these events. $\mathcal{E}_S, \mathcal{E}_T$ are independent if S, T are disjoint. A fixed progression S intersects at most $2kn$ others: choose $x \in S$ in k ways and then choose $j \in [k]$ to be the position of x in T and then choose d in at most $(n - 1)/(k - 1) \leq 2n/k$ ways. Now apply the Local Lemma.

   Some explanation: given x in position j of T, we must choose d so that $x + (k - 1 - j)d \leq n$ which implies
$$d \leq \frac{n - x}{k - 1 - j} \leq \frac{n - 1}{k - 1} \leq \frac{2n}{k}.$$
   We can improve this analysis by noting that we must also have $x - dj \geq 0$. So the number of choices for d, j is bounded by
$$\sum_{j=0}^{k-1} \min\left\{\frac{n - x}{k - 1 - j}, \frac{x}{j}\right\}. \tag{1}$$

The expression in (1) can be shown to be less than n, implying that we can replace the 8 by 4 in the statement of the question.