21-301 Combinatorics
Homework 4
Due: Monday, October 6

1. Let $\mathcal{A}$ be a family of sub-sets of $[n]$. We say that $\mathcal{A}$ is *2-secure* if there do not exist $A, B, C, D \in \mathcal{A}$ such that $A \cap B \subseteq C \cup D$. Use the probabilistic method to show the existence of a 2-secure family of exponential size.

2. Let $G = (V, E)$ be a graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose that for every v and $c \in S(v)$ there are at most d neighbors u of v such that c lies in $S(u)$. Use the local lemma to prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$. (By proper we mean that adjacent vertices get distinct colors.)

3. Show that if $8nk2^{1-k} < 1$ then one can 2-color the integers $1, 2, \ldots, n$ such that there is no mono-colored arithmetic progression of length k.
   (An arithmetic progression of length k is a set $\{a, a+d, \ldots, a+(k-1)d\}$.)