Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2025: Test 3

Name:____________________________________

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Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
Prove that if we 2-color the edges of K_4 then there is a monochromatic copy of copy of K_3 or a monoochromatic copy of P_3.
(K_3 is a triangle and $P_3 \neq K_3$ is a path with 3 edges.)
Construct an example of an edge colored K_4 where the red edges contain no K_3 and the blue edges contain no P_3.

Solution: at least 2 of the edges incident with vertex 1 must have the same color. So assume that $\{1,2\}$ and $\{1,3\}$ are red. If $\{1,4\}$ is red then either (i) $\{2,3\},\{2,4\},\{3,4\}$ are blue, giving a blue K_3 or (ii) one of these edges, say $\{2,3\}$ is red, but then there is a red P_3, $(3,2,1,4)$.
So, assume that $\{1,4\}$ is blue. Now if $\{2,3\}$ is red then there is a red K_3, $\{1,2,3\}$. So assume that $\{2,3\}$ is blue. If $\{4,3\}$ is red then there is a red P_3, $(4,3,1,2)$ and if $\{4,3\}$ is blue then there is a blue P_3, $(1,4,3,2)$.

For the second part, color $\{1,2\},\{1,3\},\{1,4\}$ red and the rest blue.

Q2: (33pts)
Determine the Sprague-Grundy numbers for the take-away game where $S = \{1, 4\}$. Give an inductive proof of your claimed answer.

Solution: the first few numbers are

j	0	1	2	3	4	5	6	7	8	9	10
$g(j)$	0	1	0	1	2	0	1	0	1	2	0

This suggests that

$$g(n) = \begin{cases} 0 & n \mod 5 = 0, 2. \\ 1 & n \mod 5 = 1, 3. \\ 2 & n \mod 5 = 4. \end{cases}$$

We verify this by induction on n, using the above table as a basis. Suppose that $n = 5m, m \geq 2$.

$$\begin{aligned}
g(5m+1) &= \text{mex}\{g(5m), g(5m-3)\} = \text{mex}\{0, 0\} = 1. \\
g(5m+2) &= \text{mex}\{g(5m+1), g(5m-2)\} = \text{mex}\{0, 0\} = 0. \\
g(5m+3) &= \text{mex}\{g(5m+2), g(5m-1)\} = \text{mex}\{0, 2\} = 1. \\
g(5m+4) &= \text{mex}\{g(5m+3), g(5m)\} = \text{mex}\{1, 0\} = 2. \\
g(5m+5) &= \text{mex}\{g(5m+4), g(5m+1)\} = \text{mex}\{2, 1\} = 0.
\end{aligned}$$

Q3: (34pts)
There are n projects and project j contributes p_j profit. Let $A = \{j : p_j > 0\}$ and $B = [n] \setminus A = \{j : p_j < 0\}$. There is a digraph $D = ([n], E)$ such that if $(i, j) \in E$ then to be feasible, project i must be completed before project j can be completed.
Consider the following network. Add edges $(s, a) : a \in A$ and $(b, t) : b \in B$ to D.

Add capacities to the network so that the capacity of an $s - t$ cut $S : \bar{S}$ is finite iff S is feasible, in which case the capacity is $\Phi(S) = \sum_{j \in A} p_j - \sum_{j \in S} p_j : S \subseteq [n]$. Why does this enable one to find the most profitable and feasible set of projects?

Solution: the edges $(s, p_j), j \in A$ are given capacity p_j. the edges $(p_j, t), j \in B$ are given capacity $-p_j$. The edges of D are given infinite capacity.
Consider the cut $(S \cup \{s\}) : (\bar{S} \cup \{t\})$. If there exists $v \in S, w \in \bar{S}$ such that $(v, w) \in E$ then the capacity of this cut is infinite. So assume now that $(v, w) \in E$ implies that either $v, w \in S$ or $v, w \in \bar{S}$. Then the capacity of this cut is

$$\left(\sum_{j \in A} p_j - \sum_{j \in S \cap A} p_j \right) + \left(\sum_{j \in S \cap B} (-p_j) \right) = \Phi(S).$$

The finiteness of the cut implies that this set of projects is feasible. The maximum flow from s to t is equal to the minimum over S of $\Phi(S)$. This equals $\sum_{j \in A} p_j$ minus the total profit if we carry out the projects in S. So to maximise profit, we find a minimum cut.