Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2025: Test 2

Name:____________________________________

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Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
Let $1 \leq k \leq n/2$ and let $\mathcal{F}$ be a Sperner family consisting of sets of size at most k. Show that $|\mathcal{F}| \leq \binom{n}{k}$.
Solution: let f_i denote the number of sets in $\mathcal{F}$ of size i. Then the LYM inequality implies that

$$\sum_{i=1}^{k} \frac{f_i}{\binom{n}{i}} \leq 1.$$

Now we have $\binom{n}{i} \leq \binom{n}{k}$ for $i \leq k$ and so

$$\sum_{i=1}^{k} \frac{f_i}{\binom{n}{k}} \leq 1,$$

giving us the result.

Q2: (33pts)
Given a graph $G = (V, E)$ with maximum degree d and a partition of V into sets $V_1, V_2, \ldots, V_m$ of size $10d$. Use the local lemma to show that there is an independent set in G with one vertex from each set in the partition.
(An independent set S, is a subset of the vertices that contains no edges.)
Solution: We can remove vertices from each V_i if needed and so we can assume w.l.o.g. that $|V_i| = 10d$ for $i = 1, 2, \ldots, r$. Choose v_i randomly from V_i for $i = 1, 2, \ldots, r$ and let $S = \{v_1, v_2, \ldots, v_r\}$. For an edge $e = \{x, y\} \in E$ we let $\mathcal{E}_e$ be the event that both $x, y \in S$. Thus $\Pr(\mathcal{E}_e) \leq p = \frac{1}{100d^2}$. An event $\mathcal{E}_e$ depends only on events $\mathcal{E}_f$ for which e and f shae a common vertex. Thus the dependency graph has degree at most $20d^2$. So, $4dp \leq \frac{80d^2}{100d^2} < 1$.

Q3: (34pts)
Let $G = (V, E)$ be a graph with $|V| = n$ and minimum degree at least $\delta \geq 10$. Show that there is a set A of size at most $\frac{2\log\delta+1}{\delta}n$ such that each vertex not in A is adjacent to at least two vertices in A.
Solution: Choose S_1 from $[n]$ by including each element independently with probability p. Then let S_2 be the set of elements, notin S_1, with at most one neighbor in S_1. Let $S = S_1 \cup S_2$. It satisfies the requirements of A. Then,

$$\mathbf{E}(|S|) \leq np + n(1-p)((1-p)^{\delta} + \delta p(1-p)^{\delta-1}) \leq np + ne^{-\delta p}(1+\delta p).$$

Putting $p = \frac{\log\delta}{\delta}$ gives

$$\mathbf{E}(|S|) \leq n\frac{\log\delta}{\delta} + \frac{n}{\delta}(1+\delta p) = \frac{2\log\delta+1}{\delta}n.$$

This implies the existence of A.