1. (a) How many strings \( a_1a_2 \cdots a_n \) of length \( n \) consisting of 0’s and 1’s have no two consecutive 1’s?

(b) How many strings \( a_1a_2 \cdots a_n \) of length \( n \) consisting of 0’s and 1’s have no three consecutive 1’s and no three consecutive 0’s?

[ Hint: define the string \( b_1b_2 \cdots b_{n-1} \) as follows: \( b_i = 1 \) iff \( a_i = a_{i+1} \).]

2. Find \( a_n \) if

\[
a_n = 6a_{n-1} + 7a_{n-2}, \quad a_0 = 2, \quad a_1 = 10.
\]

3. A row of \( n \) lightbulbs, initially all off, must be turned on. Bulb 1 can be turned on or off at any time. For \( i > 1 \), bulb \( i \) can be turned on or off only when bulb \( i - 1 \) is on and all earlier bulbs are off. Let \( a_n \) be the number of steps needed to turn all on; note that \( (a_n) \) begins \((0,1,2,5,\ldots)\). Let \( b_n \) be the number of steps to turn on bulb \( n \) for the first time. Let \( c_n \) be the number of steps until we have that bulb \( n \) is the only bulb on.

(a) Show that \( c_1 = 1 \) and \( c_n = 2c_{n-1} + 1 \), for \( n \geq 1 \).

(b) Why is \( b_{n+1} = c_n + 1 \)?

(c) Why is \( a_n = b_n + a_{n-2} \) for \( n \geq 2 \)?

(d) Solve the recurrence for \( a_n \).

[Hint: suppose that we represent the state of the \( n \) light bulbs by a \( \{0,1\} \) vector \( x \) of length \( n \) where \( x_i = 1 \) iff light-bulb \( i \) is on. Suppose that \( x(k) \) is the state after \( k \) steps. Argue that \( x(c_n - t) = x(t) + (0,0,\ldots,0,1) \) for \( t = 0,1,\ldots,c_{n-1} \).]