1. Find the set of \( P \)-positions for the take-away games with subtraction sets

(a) \( S = \{1, 3, 7\} \).
(b) \( S = \{1, 4, 6\} \).

Suppose now that there are two piles and the rules for each pile are as above. Now find the \( P \) positions for the two pile game where in one pile \( S \) is as in (a) and the other pile is as in (b).

**Solution:**

(a) The first few numbers are

\[
\begin{array}{cccccccccccc}
 j & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
 g_1(j) & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\end{array}
\]

It is apparent that \( g_1(j) = j \mod 2 \) and this follows by an easy induction: If \( j \) is even then \( j - x, x \in S \) is odd and if \( j \) is odd then \( j - x, x \in S \) is even.

(b) The first few numbers are

\[
\begin{array}{cccccccccccc}
 j & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
 g_2(j) & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 1 & 2 \\
\end{array}
\]

So, we see that the pattern 0 1 0 1 2 repeats itself. Again, induction can be used to verify that this continues indefinitely.

(c) The \( P \)-positions are those \( j, k \) for which \( g_1(j) \oplus g_2(k) = 0 \).
Thus

\[
P = \{(j = 0 \mod 2 \text{ and } k = 0, 2 \mod 5) \text{ or } (j = 1 \mod 2 \text{ and } k = 1, 3 \mod 5)\}.
\]

2. Consider the following game: There is a pile of \( n \) chips. A move consists of removing any \emph{proper} factor of \( n \) chips from the pile. (For the purposes of this question, a proper factor of \( n \), is any factor, including 1, that is strictly less than \( n \).) The player to leave a pile with one chip wins. Determine the \( N \) and \( P \) positions and a winning strategy from an \( N \) position.

**Solution:** \( n \) is a \( P \)-position iff it is odd. If \( n \) is even then the next player can simply remove one chip. If \( n \) is odd, then any factor of \( n \) is also odd.

3. In a take-away game, the set \( S \) of the possible numbers of chips to remove is finite. Show that the Sprague-Grundy numbers satisfy \( g(n) \leq |S| \) where \( n \) is the number of chips remaining.

**Solution:** Observe that for any finite set \( A \), \( \text{mex}(A) \leq |A| \) since \( \text{mex}(A) > |A| \) implies that \( A \subseteq \{0, 1, 2, \ldots, |A|\} \) which is obviously impossible. In the take-away game \( g(n) \) is the mex of a set of size at most \( |S| \) and the result follows.