Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2023: Test 4

Name:____________________________________

Andrew ID:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
Find the set of P-positions for the take-away games with subtraction sets

R_1: $S = \{2, 4\}$.

R_2: $S = \{1, 3\}$.

Suppose now that there are two piles, pile one has 13 chips and pile two has 10 chips and the rule for pile i is R_i, $i = 1, 2$. Is this a P or N position?
Solution:let g_1, g_2 denote the SG-numbers for the two games. We have

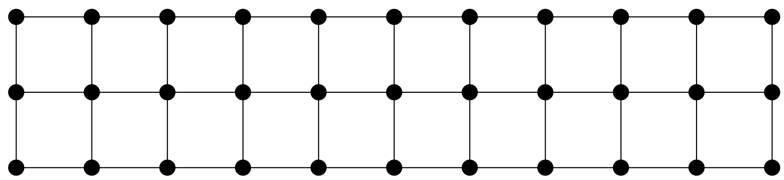
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$g_1(n)$	0	1	1	2	2	0	0	1	1	2	2	0	0	1	1	2	2	0
$g_2(n)$	0	1	0	1	2	0	1	0	1	2	0	1	0	1	2	0		

An easy induction shows that

$$g_1(n) = \begin{cases} 0 & n = 0 \text{ or } 5 \mod 6 \\ 1 & n = 1 \text{ or } 2 \mod 6 \\ 2 & n = 3 \text{ or } 4 \mod 6 \end{cases} \text{ and } g_2(n) = n \mod 2.$$

The P-positions for the two-pile game are when $g_1(n) \oplus g_2(n) = 0$. We must compute $g_1(13) \oplus g_2(10) = 1 \oplus 0 \neq 0$. So this is an N-position.

Q2: (33pts)
How many ways are there of coloring the faces of the $2 \times n$ grid when (i) there are two colors, (ii) n is even and (iii) the group acting is $e, a, b.c.$ e is the identity, a, b are reflections about the horizontal and vertical middle lines respectively and c is a rotation through $180°$.



Solution: we have

g	e	a	b	c
$\lvert Fix(g)\rvert$	2^{2n}	2^n	2^n	2^n

So the total number of colorings is

$$\frac{2^{2n} + 3 \times 2^n}{4}.$$

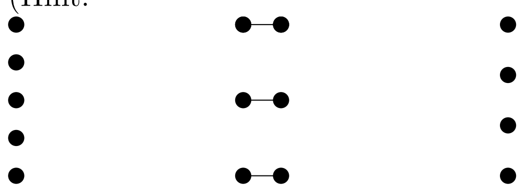
Q3: (34pts)

Let $A = \{a_1, a_2, \ldots, a_m\}$ and let $X_1, X_2, \ldots, X_M, Y_1, Y_2, \ldots, Y_N$ be subsets of A. A set of triples $\{(i_s, j_s, k_s), s = 1, 2, \ldots, m\}$ is *suitable* if (i) $i_s \neq i_t, j_s \neq j_t, k_s \neq k_t$ for $s \neq t$ and (ii) $a_{i_s} \in X_{j_s} \subseteq Y_{k_s}$ for $s = 1, 2, \ldots, m$.
For $S \subseteq [m]$, let $N_1(S) = \{j \in [M] : a_i \in X_j \text{ for some } i \in S\}$.
For $T \subseteq [M]$, let $N_2(T) = \{j \in [N] : X_i \subseteq Y_j \text{ for some } i \in T\}$.

(a) Show that if there is a suitable set of triples then

$$|N_1(S)| + |N_2(T)| \geq |S| + |T| \text{ for all } S \subseteq [m], T \subseteq N_1(S). \quad (1)$$

(b) Use the Max-Flow Min-Cut theorem to show that if (1) holds, then there is a suitable set of triples.
(Hint:



)
(You can also prove this via the use of Hall's Theorem.)
Solution: (a) if there is a suitable set $\{(i_s, j_s, k_s), s = 1, 2, \ldots, m\}$ then for $S \subseteq [m], T \subseteq N_1(S)$ we have that (i) $N_1(S) \supseteq \{j_s : i_s \in S\}$ so that $|N_1(S)| \geq |S|$ and (ii) $N_2(T) \supseteq \{k_s : j_s \in T\}$ so that $|N_2(T)| \geq |T|$.
(b) We set up a network with $2 + m + 2M + N$ vertices,

$$a_1, \ldots, a_m, X_1, X'_1 \ldots, X_M, X'_M, Y_1, \ldots, Y_N, s, t.$$

We put an edge of capacity 1 between s and $a_i, i \in [m]$ and between $Y_j, j \in [N]$ and t and between X_i and X'_i for $i \in [M]$. Then we add an edge of infinite capacity between a_i and X_j and between X'_j and Y_k, for all i, j, k.
Consider the capacity of an $s - t$ cut $X : \bar{X}$. Let $S = X \cap A$ and note that if X does not contain $N_1(S)$ then the cut has infinite capacity. So assume $N_1(S) \subseteq X$. Now let $T = X \cap \{X'_1, \ldots, X'_m\}$ and analogously, we can assume that $N_2(T) \subseteq X$. If X contains $x \in \{X_1, \ldots, X_M\} \setminus N_1(S)$ then we can remove it without changing the cut capacity. Similarly, if X contains $x' \in \{X'_1, \ldots, X'_M\}$ such that $x \notin N_1(S)$ then we can remove it without increasing cut capacity.

The capacity of such a cut is

$$m - |S| + |N_1(S)| - |T| + |N_2(T)| \geq m$$

as desired.
As an alternative proof: putting $T = \emptyset$ in (1) and applying Hall's theorem we see that there is a matching of A into $\{X_1, \ldots, X_M\}$. Suppose w.l.o.g. that this matching is $\{(a_i, X_i) : i \in [m]\}$. Putting $S = \emptyset$ in (1) and applying Hall's theorem we see that there is a matching of $\{X_1', \ldots, X_m'\}$ into $\{Y_1, \ldots, Y_M\}$. Again, w.l.o.g. we can assume that this matching is $\{(X_i', Y_i) :\in [m]\}$. In which case, $\{(i, X_i, Y_i) : i \in [m]\}$ is a suitable set of triples.