Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2023: Test 3

Name:____________________________________

Andrew ID:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
(a) Let x, y, z be integers. Show that at least one of $\frac{x+y}{2}, \frac{x+z}{2}, \frac{y+z}{2}$ is integer.
(b) Let $N = 2^n + 1$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ be lattice points in $\mathbb{R}^n$ i.e. all components of $\mathbf{x}_i$ are integer for $i \in [N]$. Show that there must be $i \neq j$ such that $\frac{\mathbf{x}_i+\mathbf{x}_j}{2}$ is also a lattice point of $\mathbb{R}^n$.
Solution: (a) By the PHP there must be 2 out of x, y, z with the same parity. Assume that x, y have the same parity. Then $x + y$ is even and $(x + y)/2$ is an integer.
(b) Define, for $i = 1, 2, \ldots, n$, the vector $\mathbf{v}_i \in \{0, 1\}^n$ by $\mathbf{v}_{i,j} = 1$ iff $\mathbf{x}_{i,j}$ is odd. Now $|\{0, 1\}^n| = 2^n < N$ and so by the PHP there must be $i \neq j$ such that $\mathbf{v}_i = \mathbf{v}_j$. But then $\mathbf{x}_i + \mathbf{x}_j$ only has even entries and $(\mathbf{x}_i + \mathbf{x}_j)/2$ is a lattice point.

Q2: (33pts)
In a town of n citizens there are m sports clubs $A_1, A_2, \ldots, A_m$ and m theater clubs $B_1, B_2, \ldots, B_m$. Show that if $|A_i \cap B_i|$ is odd for $i \in [m]$ and $|A_i \cap B_j|$ is even for $i \neq j \in [m]$ then $m \leq n$.
Solution: Define, for $i = 1, 2, \ldots, m$,the n-vector $\mathbf{a}_i \in F_2^n$ by $\mathbf{a}_{i,j} = 1$ iff $j \in A_i$. Define $\mathbf{b}_i, i = 1, 2, \ldots, m$ similarly w.r.t. the B_j. We argue that $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are linearly independent over F_2. Suppose then that $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_m\mathbf{a}_m = 0$. Then, for each i,

$$0 = (c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_m\mathbf{a}_m) \cdot \mathbf{b}_i = \sum_{j=1}^{m} c_j\mathbf{a}_j \cdot \mathbf{b}_i = c_i.$$

This proves the linear independence of the $\mathbf{a}_i$.

Q3: (34pts)
Let $r_k = R(3, 3, \ldots, 3; 2)$ be the smallest integer n such that if the edges of K_n are colored using $k \geq 2$ colors then there is a monochromatic triangle. Show that $r_k \leq k(r_{k-1} - 1) + 2$.
Solution: let $N = k(r_{k-1} - 1) + 2$. For $i = 1, 2, \ldots, k$, let $V_i = \{j \geq 2 : c(\{1, j\}) = i$ be the set of vertices connected to vertex 1 by an edge of color i.
Because $|V_1| + \cdots + |V_k| = k(r_{k-1} - 1) + 1$, there must be some i such that $|V_i| \geq r_{k-1}$.
If V_i contains an edge $\{x, y\}$ of color i then the triangle $\{1, x, y\}$ is monochromatic, all 3 edges being of color i. Otherwise the edges of V_i have been colored with $k-1$ colors. Because $|V_i| \geq r_{k-1}$ there will be a mono-chromatic triangle in V_i.