Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2023: Test 3

Name:_______________________________

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Q1: (33pts)
(a) Let \(x, y, z\) be integers. Show that at least one of \(\frac{x+y}{2}, \frac{x+z}{2}, \frac{y+z}{2}\) is integer.

(b) Let \(N = 2^n + 1\) and \(x_1, x_2, \ldots, x_N\) be lattice points in \(\mathbb{R}^n\) i.e. all components of \(x_i\) are integer for \(i \in [N]\). Show that there must be \(i \neq j\) such that \(\frac{x_i + x_j}{2}\) is also a lattice point of \(\mathbb{R}^n\).

Solution: (a) By the PHP there must be 2 out of \(x, y, z\) with the same parity. Assume that \(x, y\) have the same parity. Then \(x + y\) is even and \((x + y)/2\) is an integer.

(b) Define, for \(i = 1, 2, \ldots, n\), the vector \(v_i \in \{0, 1\}^n\) by \(v_{i,j} = 1\) iff \(x_{i,j}\) is odd. Now \(|\{0, 1\}^n| = 2^n < N\) and so by the PHP there must be \(i \neq j\) such that \(v_i = v_j\). But then \(x_i + x_j\) only has even entries and \((x_i + x_j)/2\) is a lattice point.
Q2: (33pts)
In a town of \( n \) citizens there are \( m \) sports clubs \( A_1, A_2, \ldots, A_m \) and \( m \) theater clubs \( B_1, B_2, \ldots, B_m \). Show that if \( |A_i \cap B_i| \) is odd for \( i \in [m] \) and \( |A_i \cap B_j| \) is even for \( i \neq j \in [m] \) then \( m \leq n \).

**Solution:** Define, for \( i = 1, 2, \ldots, m \), the \( n \)-vector \( a_i \in F_2^n \) by \( a_{i,j} = 1 \) iff \( j \in A_i \). Define \( b_i, i = 1, 2, \ldots, m \) similarly w.r.t. the \( B_j \). We argue that \( a_1, a_2, \ldots, a_m \) are linearly independent over \( F_2 \). Suppose then that \( c_1a_1 + c_2a_2 + \cdots + c_ma_m = 0 \). Then, for each \( i \),

\[
0 = (c_1a_1 + c_2a_2 + \cdots + c_ma_m) \cdot b_i = \sum_{j=1}^{m} c_j a_{j,i} \cdot b_i = c_i.
\]

This proves the linear independence of the \( a_i \).
Q3: (34pts)
Let \( r_k = R(3, 3, \ldots, 3; 2) \) be the smallest integer \( n \) such that if the edges of \( K_n \) are colored using \( k \geq 2 \) colors then there is a monochromatic triangle. Show that \( r_k \leq k(r_{k-1} - 1) + 2 \).

Solution: let \( N = k(r_{k-1} - 1) + 2 \). For \( i = 1, 2, \ldots, k \), let \( V_i = \{ j \geq 2 : c(\{1, j\}) = i \} \) be the set of vertices connected to vertex 1 by an edge of color \( i \).

Because \( |V_1| + \cdots + |V_k| = k(r_{k-1} - 1) + 1 \), there must be some \( i \) such that \( |V_i| \geq r_{k-1} \).

If \( V_i \) contains an edge \( \{x, y\} \) of color \( i \) then the triangle \( \{1, x, y\} \) is monochromatic, all 3 edges being of color \( i \). Otherwise the edges of \( V_i \) have been colored with \( k-1 \) colors. Because \( |V_i| \geq r_{k-1} \) there will be a mono-chromatic triangle in \( V_i \).