Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2023: Test 2

Name:___________________________________

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Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
Let $G = (V, E)$ be a graph on n vertices, with minimum degree $d > 1$. $S \subseteq V$ is a 2-dominating set if $v \notin S$ implies that v has at least two neighbors in S. Show that G contains a 2-dominating set of size at most $\frac{3n \log d}{d}$.
(Hint: $p = \frac{\log d}{d}$.)
Solution: Place each vertex of V independently into $S_1 \subseteq V$ with probability $p = \frac{\log d}{d}$. Let S_2 be the set of vertices not in S_1 with at most one neighbor in S_1. Then $S = S_1 \cup S_2$ is a 2-dominating set. We have

$$\begin{aligned}
\mathbf{E}(|S|) &\leq np + n(1-p)((1-p)^d + dp(1-p)^{d-1}) \\
&\leq n\left(\frac{\log d}{d} + e^{-\log d} + \log d \cdot e^{-\log d}\right) \\
&\leq \frac{3n \log d}{d}.
\end{aligned}$$

Q2: (33pts)
Let $G = (V, E)$ be a graph with maximum degree $d > 1$. Show that the vertices of G can be colored with $q \geq 10d^4$ colors such that no color appears twice in the neighborhood of a vertex.
Solution: randomly color each vertex. Let $\mathcal{E}_v$ denote the event that v's neigborhood contains a repeated color. Then

$$p = \Pr(\mathcal{E}_v) \leq \mathbf{E}(\text{number of repeated colors in } v\text{'s neighborhood}) = \frac{\binom{d}{2}}{q} \leq \frac{d^2}{2q}.$$

Events $\mathcal{E}_v, \mathcal{E}_w$ are dependent only if they share a common vertex. Hence the dependency graph has maximum degree d^2. Now $4d^2 \cdot d^2/q < 1$ and so the local lemma implies the result.

Q3: (34pts)
A family of sets $\mathcal{A} \subseteq \binom{[n]}{r}, r \geq 2$ satisfies,

$$A, B \in \mathcal{A},\ A \neq B \text{ implies } |A \cap B| \leq 1.$$

Let $f(n,r)$ denote the maximum size of such a family.
(a) Prove that $f(n,r) = 1$ for $r \leq n \leq 2r-2$.
(b) Prove that $f(n,r) \leq \frac{n-1}{r-1} + f(n-1,r)$ for $n \geq 2r-1$.
(c) Deduce that $f(n,r) \leq \frac{(n-r)(n+r-1)}{2(r-1)} + 1$.
(This bound is not tight. Do not worry if you get something slightly better.)
Solution:
(a) This follows from the fact that if $|A| = |B| = r$ and $|A \cap B| \leq 1$ then $|A \cup B| \geq 2r-1$.
(b) The set $\mathcal{A}$ contains at most $\frac{n-1}{r-1}$ sets that contain n. This is because these sets form a disjoint family of $(r-1)$-sets. There are at most $f(n-1,r)$ sets in $\mathcal{A}$ that do not contain n.
(c) It follows from (b) that

$$f(n,r) \leq \frac{n-1}{r-1} + \frac{n-2}{r-1} + \cdots + \frac{r-1}{r-1} + 1 \leq \frac{(n-r)(n+r-1)}{2(r-1)} + 1.$$