Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2023: Test 1

Name: ____________________________

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Q1: (33pts)
Find a recurrence for the number of sequences in \( \{a, b, c\}^n \) that do not contain \( \text{aaa} \) as a subsequence.

DO NOT SOLVE THE RECURRENCE.

Solution: Let \( B_n \) be the set of sequences in question. Let \( B_n^x, x = a, b, c \) be the sequences in \( B_n \) that begin with \( x \). Then

\[
|B_n| = |B_n^{(a)}| + |B_n^{(b)}| + |B_n^{(c)}| = |B_n^{(a)}| + 2|B_{n-1}|.
\]

Similarly,

\[
|B_n^{(a)}| = |B_n^{(aa)}| + |B_n^{(ab)}| + |B_n^{(ac)}| = |B_n^{(aa)}| + 2|B_{n-2}| = 2|B_{n-3}| + 2|B_{n-2}|.
\]

So, the recurrence for \( b_n = |B_n| \) is that

\[
b_n = 2b_{n-1} + 2b_{n-2} + 2b_{n-3}.
\]
Q2: (33pts)
The sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies the following: $a_0 = 1, a_1 = 9$ and

$$a_n = 6a_{n-1} - 9a_{n-2}$$

for $n \geq 2$.

(a): Find the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$.

(b): Find an expression for $a_n, n \geq 0$.

Solution: Multiplying the equation by $x^n$ and summing over $n \geq 2$ we obtain

$$(a(x) - 9x - 1) - (6x(a(x) - 1) + (9x^2 a(x))) = 0$$

or $a(x) = \frac{3x + 1}{(1 - 3x)^2}$.

Now

$$\frac{3x}{(1 - 3x)^2} = \sum_{n=0}^{\infty} n3^n$$

and

$$\frac{1}{(1 - 3x)^2} = \sum_{n=0}^{\infty} (n + 1)3^n.$$  

So

$$a_n = (2n + 1)3^n.$$
Q3: (34pts)
A table has 4n seats. n families sit round the table, so that clockwise we have Man, Woman, Boy, Girl. How many ways are there of seating people so that no family sits completely together?

Solution: Let $A_i$ be those sittings where family $i$ sits together. Then if $|S| = s$,

$$|A_S| = \binom{n}{s} s!(n-s)!^4.$$

Explanation: $\binom{n}{s}$ choices of places to seat $S$ and then $s!$ orderings of the families and $(n-s)!^4$ ways of arranging the rest of the people. Thus, the number of possible seatings is

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s}^2 s!(n-s)!^4.$$