1. Let $r_n = r(3, 3, \ldots, 3)$ be the minimum integer such that if we $n$-color the edges of the complete graph $K_N$ there is a monochromatic triangle.

(a) Show that $r_n \leq n(n-1) + 2$.

(b) Using $r_2 = 6$, show that $r_n \leq \left\lfloor \frac{n(n-1)}{e} \right\rfloor + 1$.

**Solution:** Let $N = n(n-1) + 2$ and consider an $n$-coloring $\sigma$ of the edges of $K_N$. Now consider the $N-1$ edges incident to vertex $N$. There must be a color, $n$ say, that is used at least $r_n-1$ times, Pigeon Hole Principle. Now let $V \subseteq [N-1]$ denote the set of vertices $v$ for which the edge $\{v, N\}$ is colored $n$. Consider the coloring of the edges of $V$ induced by $\sigma$. If one of these $\{v_1, v_2\}$ has color $N$ then it makes a triangle $v_1, v_2, N$ with 3 edges colored $n$. Otherwise the edges of $V$ only use $n-1$ colors and since $|V| \geq r_n-1$ we see by induction that $V$ contains a mono-chromatic triangle.

**Solution:** Divide the inequality (a) by $n!$ and putting $s_n = r_n/n!$ we obtain

$$s_n \leq s_{n-1} - \frac{1}{(n-1)!} + \frac{2}{n!}.$$  \hspace{1cm} (1)

We write this as

$$s_n - s_{n-1} \leq -\frac{1}{(n-1)!} + \frac{2}{n!}$$

$$s_{n-1} - s_{n-2} \leq -\frac{1}{(n-2)!} + \frac{2}{(n-1)!}$$

$$\vdots$$

$$s_3 - s_2 \leq -\frac{1}{2!} + \frac{2}{3!}$$

Summing gives

$$s_n - s_2 \leq -\frac{1}{2!} + \frac{1}{n!} + \sum_{k=3}^{n} \frac{1}{k!} \leq \frac{1}{n!} + e - 2.$$ 

Now $s_2 = 3$ and multiplying the above by $n!$ gives $r_n \leq n!e + 1$. We round down, as $r_n$ is an integer.

2. Show that $r(C_4, C_4) = 6$, where $C_4$ denotes a cycle of length 4.

**Solution:** (a) Color the edges of the 5-cycle $(1,2,3,4,5,1)$ Red and the edges of the remaining 5-cycle $(1,3,5,2,4,1)$ Blue. There are no mono-chromatic 4-cycles.

(b) Each vertex is incident with at least 3 edges of the same color. So, we can assume that 1,2,3 each have at least 3 red neighbors $N_1, N_2, N_3$. If $N_1, N_2 \subseteq \{3, 4, 5, 6\}$ then $|N_1 \cap N_2| \geq 2$ and then there is a $C_4$ containing vertices 1,2.

We can assume then that 1,2,3 form a red triangle. If $4 \in N_1 \cap N_2$ then we have that 1,3,2,4,1 is a red $C_4$. 


So we can assume that \(|N_i \cap N_j| = 1\) for all \(i, j\) and that \(N_1 = \{2, 3, 4\}, N_2 = \{1, 3, 5\}, N_3 = \{1, 2, 6\}\). If \(\{4, 5\}\) is red then \(1, 2, 5, 4, 1\) is a red \(C_4\). So we can assume that \(4, 5, 6\) form a blue triangle.

If \(\{1, 5\}\) is red then \(1, 5, 3, 2, 1\) is a red \(C_4\). So we can assume that \(\{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 6\}, \{3, 4\}, \{3, 5\}\) are all blue.

But then \(1, 5, 4, 6, 1\) is a blue \(C_4\).

3. Use Dilworth’s theorem to show that if in a bipartite graph \(G = (A, B, E)\) we have that \(|N(S)| \geq |S| - t\) for all \(S \subseteq A\), then there is a matching of size at least \(|A| - t\).

**Solution:** Let \(G = (A \cup B, E)\) be a bipartite graph which satisfies the given condition. Define a poset \(P = A \cup B\) and define \(<\) by \(a < b\) only if \(a \in A, b \in B\) and \((a, b) \in E\). Suppose that the largest anti-chain in \(P\) is \(A = \{a_1, a_2, \ldots, a_h, b_1, b_2, \ldots, b_k\}\) and let \(s = h + k\).

Now

\[N(\{a_1, a_2, \ldots, a_h\}) \subseteq B \setminus \{b_1, b_2, \ldots, b_k\}\]

for otherwise \(A\) will not be an anti-chain. From the given condition we see that

\[|B| - k \geq h - t\]

or equivalently \(|B| \geq s - t\).

Now by Dilworth’s theorem, \(P\) is the union of \(s\) chains: A matching \(M\) of size \(m\), \(|A| - m\) members of \(A\) and \(|B| - m\) members of \(B\). But then

\[m + (|A| - m) + (|B| - m) = s \leq |B| + t\]

and so \(m \geq |A| - t\).