1. How many sequences \((a_1, a_2, \ldots, a_m) \in [n]^m\) satisfy \(a_1 < a_2 < \cdots < a_m\)? How many satisfy \(a_1 \leq a_2 \leq \cdots \leq a_m\)?

**Solution:** The sequence \(a_1 < a_2 < \cdots < a_m\) defines a subset \(\{a_1, a_2, \ldots, a_m\}\) of \([n]\) and conversely, every subset of size \(m\) defines a sequence after ordering the elements. Thus the answer to the first part is \(\binom{n}{m}\).

For the second part, let \(a_0 = 0\) and \(a_{m+1} = n\) and let \(b_i = a_i - a_{i-1}\) for \(i = 1, 2, \ldots, m+1\). There is a 1-1 correspondence between the sequences \(a_0 = 0, a_1, a_2, \ldots, a_m, a_{m+1} = n\) and the sequences \(b_1, b_2, \ldots, b_{m+1}\) since we can recover \(a_i = b_1 + b_2 + \cdots + b_i\) for \(i = 1, 2, \ldots, m\). Now we have \(b_1 \geq 1\) and \(b_i \geq 0, i = 2, 3, \ldots, m+1\) and \(b_1 + b_2 + \cdots + b_{m+1} = n\) and any sequence \(b_1, b_2, \ldots, b_{m+1}\) with these properties gives rise to a sequence \(a_1 \leq a_2 \leq \cdots \leq a_m\). Thus there are \(\binom{n-1}{m} + \binom{m+1}{m} - 1\) such sequences.

2. Suppose that a round table has \(3n\) seats. Suppose that \(n\) families arrive consisting of man/woman/child. They are to be seated round the table in triples: adult, child, adult. How many ways of seating the guests are there so that no family sits together as a complete triple of adult, child, adult.

**Solution:** This is a variation on the “Problème des Ménages”. Let \(A_i\) denote the set of seatings in which family \(i\) sit together. Then for \(|S| = k\) we have

\[
|A_S| = 3k!2^k(2(n-k))!(n-k)! \binom{n}{k} = 3n!(2(n-k))!2^k.
\]

**Explanation:** 3 choices for where to place the “first” child. Given this, we choose \(k\) seats for children in \(\binom{n}{k}\) ways. Then we order the children of the \(k\) families in \(k!\) ways. Then we place their parents next to them in \(2^k\) ways. We place the remaining children in \((n-k)!\) ways and adults in \((2(n-k))!\) ways.

Putting it all together we get that the number of seating arrangements is

\[
3n! \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2(n-k))!2^k.
\]

3. Suppose that we have \(2n\) distinguishable balls. Suppose that there are \(n\) colors and that we have 2 balls of each color. How many ways are there of placing the balls into \(n\) boxes, two balls per box, so that there are exactly \(k\) boxes containing balls of the same color?

**Solution:** Let \(A_i\) denote the set of allocations in which box \(i\) gets two balls of the same color. Arguing as in the notes on Scrambled Allocations, we see that if \(|S| = \ell\) then

\[
|A_S| = \frac{(2(n-\ell))!}{2^{n-\ell}} \times \binom{n}{\ell} \times \ell!.
\]

The extra factor \(\binom{n}{\ell} \times \ell!\) comes from allocating the colors to the boxes.
Applying the inclusion-exclusion formula for elements in exactly $k$ of the $A_i$, we get the expression

$$\sum_{\ell=k}^{n} \binom{n}{\ell} (-1)^{\ell-k} \binom{\ell}{k} \frac{(2(n-\ell))!}{2^{2n-\ell}} \binom{n}{\ell} \ell!.$$