Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2010: Test 4

Name: ____________________________

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Q1: (33pts)
Let \( k \geq 1 \) be a positive integer. Consider the one pile take-away game \( G_k \) where a player is allowed to remove between 1 and \( k \) chips in a move. Show that the Grundy numbers \( g_k \) are given by \( g_k(n) = n \mod (k + 1) \).

Suppose that 11,24 is the position in a game of \( G_5 \oplus G_3 \). Is this an N or P position?

**Solution** We prove this by induction on \( n \). This is true for \( n = 0 \). If \( 1 \leq n \leq k \) then using induction

\[
g(n) = \text{mex}\{g(0), g(1), \ldots, g(n-1)\} = \text{mex}\{0, 1, \ldots, n-1\} = n = n \mod (k+1).
\]

If \( n > k \) then \( n = a(k + 1) + b \) for integers \( a \geq 1, 0 \leq b \leq k \). So,

\[
g(n) = \text{mex}\{g(n-k), g(n-k+1), \ldots, g(n-1)\}
\]

\[
= \text{mex}\{n-k \mod (k+1), \ldots, n-1 \mod (k+1)\}
\]

\[
= \text{mex}\{b-k \mod (k+1), \ldots, b-1 \mod (k+1)\}
\]

\[
= \text{mex}\{b+1 \mod (k+1), \ldots, k+1 \mod (k+1), b+k \mod (k+1)\}
\]

\[
= \text{mex}\{b+1, \ldots, k, 0, 1, \ldots, b-1\}
\]

\[
= b
\]

\[
= n \mod (k+1)
\]
Q2: (33pts)
Prove that every partition matroid is also a transversal matroid.

Solution Suppose that the partition matroid $\mathcal{M}$ is defined by the disjoint sets $E_1, E_2, \ldots, E_m$ and that $I \subseteq E = \bigcup_i E_i$ is independent iff $|I \cap E_i| \leq k_i, i = 1, 2, \ldots, m$. Let $G_i$ be the complete bipartite graph with vertex set $K_i, E_i$ where $|K_i| = k_i$. Let $G = G_1 \cup G_2 \cup \cdots G_m$. A matching of $G$ is obtained by choosing at most $k_i$ from each $E_i$ and then joining each chosen vertex to a distinct vertex in $K_i$. So there is a 1-1 correspondence between matched subsetets of $E$ and independent sets in $\mathcal{M}$.
Q3: (34pts) Let $G = (V, E)$ be a connected graph and suppose that the edges $E$ are colored Red and Blue. Let $E_R$ denote the set of Red edges. Let $k \geq 1$ be an integer. Show that $G$ contains a spanning tree with at least $k$ Red edges if and only if for all $S \subseteq E$ we have

$$\min\{n - k - 1 + |\bar{S} \cap E_R|, |\bar{S}|\} \geq \kappa(S) - 1. \quad (1)$$

Here $\bar{S} = E \setminus S$ and $\kappa(S)$ is the number of components in the subgraph $G_S = (V, S)$.

**Solution** Suppose that $|V| = n$. We seek a set of edges $E$ of size $|n - 1$ that is independent in the cycle matroid $M_1$ of $G$ and the partition matroid $M_2$ where a set is independent iff it contains at most $n - k - 1$ Blue edges. Applying Edmond’s theorem we see that this exists iff

$$\min_{S \subseteq E}\{n - \kappa(S) + |\bar{S} \cap E_R|\} \geq n - 1.$$

(1) follows by subtracting $n - \kappa(S)$ from both sides.