Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2010: Test 4

Name:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
Let $k \geq 1$ be a positive integer. Consider the one pile take-away game G_k where a player is allowed to remove between 1 and k chips in a move. Show that the Grundy numbers g_k are given by $g_k(n) = n \mod (k+1)$.
Suppose that 11,24 is the position in a game of $G_5 \oplus G_3$. Is this an N or P position?
Solution We prove this by induction on n. This is true for $n = 0$. If $1 \leq n \leq k$ then using induction

$$g(n) = \text{mex}\{g(0), g(1), \ldots, g(n-1)\} = \text{mex}\{0, 1, \ldots, n-1\} = n = n \mod k+1.$$

If $n > k$ then $n = a(k+1) + b$ for integers $a \geq 1, 0 \leq b \leq k$. So,

$$\begin{aligned}
g(n) &= \text{mex}\{g(n-k), g(n-k+1), \ldots, g(n-1)\} \\
&= \text{mex}\{n-k \mod k+1, \ldots, n-1 \mod k+1\} \\
&= \text{mex}\{b-k \mod k+1, \ldots, b-1 \mod k+1\} \\
&= \text{mex}\{b+1 \mod k+1, \ldots, k+1 \mod k+1, \ldots, b+k \mod k+1\} \\
&= \text{mex}\{b+1, \ldots, k, 0, 1, \ldots, b-1\} \\
&= b \\
&= n \mod k+1
\end{aligned}$$

Q2: (33pts)

Prove that every partition matroid is also a transversal matroid.

Solution Suppose that the partition matroid $\mathcal{M}$ is defined by the disjoint sets $E_1, E_2, \ldots, E_m$ and that $I \subseteq E = \bigcup_i E_i$ is independent iff $|I \cap E_i| \leq k_i, i = 1, 2, \ldots, m$. Let G_i be the complete bipartite graph with vertex set K_i, E_i where $|K_i| = k_i$. Let $G = G_1 \cup G_2 \cup \cdots G_m$. A matching of G is obtained by choosing at most k_i from each E_i and then joining each chosen vertex to a distinct vertex in K_i. So there is a 1-1 correspondence between matched subsetets of E and independent sets in $\mathcal{M}$.

Q3: (34pts) Let $G = (V, E)$ be a connected graph and suppose that the edges E are colored Red and Blue. Let E_R denote the set of Red edges. Let $k \geq 1$ be an integer. Show that G contains a spanning tree with at least k Red edges if and only if for all $S \subseteq E$ we have

$$\min\{n - k - 1 + |\bar{S} \cap E_R|, |\bar{S}|\} \geq \kappa(S) - 1. \tag{1}$$

Here $\bar{S} = E \setminus S$ and $\kappa(S)$ is the number of components in the subgraph $G_S = (V, S)$.

Solution Suppose that $|V| = n$. We seek a set of edges E of size $|n - 1$ that is independent in the cycle matroid $\mathcal{M}_1$ of G and the partition matroid $\mathcal{M}_2$ where a set is independent iff it contains at most $n - k - 1$ Blue edges. Applying Edmond's theorem we see that this exists iff

$$\min_{S \subseteq E}\{n - \kappa(S) + |\bar{S} \cap E_R)| + \min\{n - k - 1, |\bar{S} \cap E_B|\}\} \geq n - 1.$$

(1) follows by subtracting $n - \kappa(S)$ from both sides.