Department of Mathematics  
Carnegie Mellon University

21-301 Combinatorics, Fall 2021: Test 1

Name:______________________________

Andrew ID:__________________________

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Q1: (33pts)  
How many sequences \( a_1, a_2, \ldots, a_m \in [n] \) satisfy \( a_i - a_{i-1} \geq k \) for \( 2 \leq i \leq m \), where \( k \) is a positive integer?  
(Hint: consider the sequence \( x_i = a_i - a_{i-1}, 1 \leq i \leq m + 1 \), where \( a_0 = 0 \) and \( a_{m+1} = n \).)  
Solution: following the hint we see that \( x_1 + \cdots + x_{m+1} = x_{m+1} - x_0 = n \) and \( x_i \geq k \) for \( 2 \leq i \leq m \). Then put \( y_i = x_i - k, 2 \leq i \leq m, y_1 = x_1 - 1, y_{m+1} = x_{m+1} \) so that \( y_1, \ldots, y_{m+1} \geq 0 \) and \( y_1 + \cdots + y_{m+1} = n - (m - 1)k - 1 \). There is a 1-1 correspondence between the \( a_i \)'s and the \( y_i \)'s and so the answer is \( \binom{n-(m-1)k-1+m}{m-1} \).
Q2: (33pts)
The sequence \(a_0, a_1, \ldots, a_n, \ldots\) satisfies the following: \(a_0 = 1, a_1 = 9\) and
\[
a_n = 4a_{n-1} - 4a_{n-2}
\]
for \(n \geq 2\).

(a): Find the generating function \(a(x) = \sum_{n=0}^{\infty} a_n x^n\).

(b): Find an expression for \(a_n, n \geq 0\).

Solution:

\[
\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} (4a_{n-1} - 4a_{n-2}) x^n
\]
\[
a(x) - 9x - 1 = 4x(a(x) - 1) - 4x^2 a(x)
\]
\[
a(x) = \frac{5x + 1}{4x^2 - 4x + 1} = \frac{5x + 1}{(1 - 2x)^2}
\]
\[
a(x) = \sum_{n=0}^{\infty} \binom{n+1}{1} (2x)^n + 5x \sum_{n=0}^{\infty} \binom{n+1}{1} (2x)^n
\]
\[
a_n = (n + 1)2^n + 5n2^{n-1} = 2^n + 7n2^{n-1}
\]
Q3: (34pts)
n children take off their jackets and shoes and put them into a pile on the floor and go and play. How many ways are there of giving to each of the children a pair of matching shoes and a jacket, so that no child gets his/her own jacket and shoes.

Solution: Suppose that child $i$ is given the jacket of child $\pi_1(i)$ and the shoes of child $\pi_2(i)$. Let

$$A_i = \{ (\pi_1, \pi_2) : \pi_1(i) = \pi_2(i) = i \}$$

for $i = 1, 2, \ldots, n$.

We need to compute $|\bigcap_{i=1}^n \bar{A}_i|$. Now if $|S| = k$ then $|A_S| = ((n-k)!)^2$ since we have fixed $\pi_1(i), \pi_2(i)$ for $i \in S$ and the remaining values can be permuted arbitrarily. Thus

$$\left| \bigcap_{i=1}^n \bar{A}_i \right| = \sum_{S \subseteq [N]} (-1)^{|S|} ((n-|S|)!)^2 = \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k!)^2.$$