Covered so far
8/30/2021
Let $\phi(m, n)$ be the number of mappings from $[n]$ to $[m]$.

**Theorem**

$$\phi(m, n) = m^n$$

**Proof**  
By induction on $n$.  

$$\phi(m, 0) = 1 = m^0.$$ 

$$\phi(m, n + 1) = m\phi(m, n)$$ 
$$= m \times m^n$$ 
$$= m^{n+1}.$$ 

$\phi(m, n)$ is also the number of sequences $x_1x_2 \cdots x_n$ where
Let $\psi(n)$ be the number of subsets of $[n]$.

**Theorem**

$\psi(n) = 2^n$.

**Proof**  
(1) By induction on $n$.  
$\psi(0) = 1 = 2^0$.  

$\psi(n+1)$  
$= \#\{\text{sets containing } n+1\} + \#\{\text{sets not containing } n+1\}$  
$= \psi(n) + \psi(n)$  
$= 2^n + 2^n$  
$= 2^{n+1}$.  

Covered so far
There is a general principle that if there is a 1-1 correspondence between two finite sets $A, B$ then $|A| = |B|$. Here is a use of this principle.

**Proof** (2).

For $A \subseteq [n]$ define the map $f_A : [n] \rightarrow \{0, 1\}$ by

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \not\in A \end{cases}.$$

$f_A$ is the characteristic function of $A$.

Distinct $A$’s give rise to distinct $f_A$’s and vice-versa.

Thus $\psi(n)$ is the number of choices for $f_A$, which is $2^n$ by Theorem 51. □
Let $\psi_{\text{odd}}(n)$ be the number of odd subsets of $[n]$ and let $\psi_{\text{even}}(n)$ be the number of even subsets.

**Theorem**

$$\psi_{\text{odd}}(n) = \psi_{\text{even}}(n) = 2^{n-1}.$$ 

**Proof**

For $A \subseteq [n-1]$ define

$$A' = \begin{cases} A & |A| \text{ is odd} \\ A \cup \{n\} & |A| \text{ is even} \end{cases}$$

The map $A \to A'$ defines a bijection between $[n-1]$ and the odd subsets of $[n]$. So $2^{n-1} = \psi(n-1) = \psi_{\text{odd}}(n)$. Furthermore,

$$\psi_{\text{even}}(n) = \psi(n) - \psi_{\text{odd}}(n) = 2^n - 2^{n-1} = 2^{n-1}.$$
Let $\phi_{1-1}(m, n)$ be the number of 1-1 mappings from $[n]$ to $[m]$.

**Theorem**

$$\phi_{1-1}(m, n) = \prod_{i=0}^{n-1} (m - i). \quad (1)$$

**Proof** Denote the RHS of (1) by $\pi(m, n)$. If $m < n$ then $\phi_{1-1}(m, n) = \pi(m, n) = 0$. So assume that $m \geq n$. Now we use induction on $n$.

If $n = 0$ then we have $\phi_{1-1}(m, 0) = \pi(m, 0) = 1$.

In general, if $n < m$ then

$$\phi_{1-1}(m, n + 1) = (m - n)\phi_{1-1}(m, n)$$
$$= (m - n)\pi(m, n)$$
$$= \pi(m, n + 1).$$
\( \phi_{1-1}(m, n) \) also counts the number of length \( n \) ordered sequences distinct elements taken from a set of size \( m \).

\[
\phi_{1-1}(n, n) = n(n-1) \cdots 1 = n!
\]

is the number of ordered sequences of \([n]\) i.e. the number of permutations of \([n]\).
Binomial Coefficients

\[
\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1) \cdots (n-k+1)}{k(k-1) \cdots 1}
\]

Let \(X\) be a finite set and let \(\binom{X}{k}\) denote the collection of \(k\)-subsets of \(X\).

**Theorem**

\[\left|\binom{X}{k}\right| = \binom{|X|}{k}.\]

**Proof** Let \(n = |X|\),

\[k! \left|\binom{X}{k}\right| = \phi_{1-1}(n, k) = n(n-1) \cdots (n-k+1).\]
Let $m, n$ be non-negative integers. Let $\mathbb{Z}_+$ denote the non-negative integers. Let

$$S(m, n) = \{(i_1, i_2, \ldots, i_n) \in \mathbb{Z}_+^n : i_1 + i_2 + \cdots + i_n = m\}.$$  

\textbf{Theorem}

$$|S(m, n)| = \binom{m + n - 1}{n - 1}.$$  

\textbf{Proof} imagine $m + n - 1$ points in a line. Choose positions $p_1 < p_2 < \cdots < p_{n-1}$ and color these points red. Let $p_0 = 0, p_n = m + 1$. The gap sizes between the red points

$$i_t = p_t - p_{t-1} - 1, \quad t = 1, 2, \ldots, n$$

form a sequence in $S(m, n)$ and vice-versa.  

\[\square\]
$|S(m, n)|$ is also the number of ways of coloring $m$ indistinguishable balls using $n$ colors.

Suppose that we want to count the number of ways of coloring these balls so that each color appears at least once i.e. to compute $|S(m, n)^*|$ where, if $N = \{1, 2, \ldots, \}$

$$S(m, n)^* = \{(i_1, i_2, \ldots, i_n) \in N^n : i_1 + i_2 + \cdots + i_n = m\}$$

$$= \{(i_1 - 1, i_2 - 1, \ldots, i_n - 1) \in \mathbb{Z}_+^n : (i_1 - 1) + (i_2 - 1) + \cdots + (i_n - 1) = m - n\}$$

Thus,

$$|S(m, n)^*| = \binom{m - n + n - 1}{n - 1} = \binom{m - 1}{n - 1}.$$
How many ways (patterns) are there of placing $k$ 1’s and $n - k$ 0’s at the vertices of a polygon with $n$ vertices so that no two 1’s are adjacent? Choose a vertex $v$ of the polygon in $n$ ways and then place a 1 there. For the remainder we must choose $a_1, \ldots, a_k \geq 1$ such that $a_1 + \cdots + a_k = n - k$ and then go round the cycle (clockwise) putting $a_1$ 0’s followed by a 1 and then $a_2$ 0’s followed by a 1 etc..
Each pattern $\pi$ arises $k$ times in this way. There are $k$ choices of $v$ that correspond to a 1 of the pattern. Having chosen $v$ there is a unique choice of $a_1, a_2, \ldots, a_k$ that will now give $\pi$.

There are $\binom{n-k-1}{k-1}$ ways of choosing the $a_i$ and so the answer to our question is

$$\frac{n(n-k-1)}{k(k-1)}.$$
Theorem

Symmetry

\[
\binom{n}{r} = \binom{n}{n-r}
\]

Proof  Choosing \( r \) elements to include is equivalent to choosing \( n - r \) elements to exclude.  

\( \Box \)
Theorem

Pascal’s Triangle

\[ \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \]

Proof   A \( k + 1 \)-subset of \([n + 1]\) either

(i) includes \( n + 1 \) —— \( \binom{n}{k} \) choices or

(ii) does not include \( n + 1 \) —— \( \binom{n}{k+1} \) choices.
Pascal’s Triangle

The following array of binomial coefficients, constitutes the famous triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

...
Theorem

\[
\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.
\] (2)

Proof 1: Induction on \( n \) for arbitrary \( k \).

Base case: \( n = k \); \( \binom{k}{k} = \binom{k+1}{k+1} \)

Inductive Step: assume true for \( n \geq k \).

\[
\sum_{m=k}^{n+1} \binom{m}{k} = \sum_{m=k}^{n} \binom{m}{k} + \binom{n+1}{k}
\]

\[
= \binom{n+1}{k+1} + \binom{n+1}{k} \quad \text{Induction}
\]

\[
= \binom{n+2}{k+1} \quad \text{Pascal’s triangle}
\]
Proof 2: Combinatorial argument.
If $S$ denotes the set of $k + 1$-subsets of $[n + 1]$ and $S_m$ is the set of $k + 1$-subsets of $[n + 1]$ which have largest element $m + 1$ then

- $S_k, S_{k+1}, \ldots, S_n$ is a partition of $S$.
- $|S_k| + |S_{k+1}| + \cdots + |S_n| = |S|$.
- $|S_m| = \binom{m}{k}$.
Theorem

Vandermonde’s Identity

\[ \sum_{r=0}^{k} \binom{m}{r} \binom{n}{k-r} = \binom{m+n}{k}. \]

Proof

Split \([m + n]\) into \(A = [m]\) and \(B = [m + n] \setminus [m]\). Let \(S\) denote the set of \(k\)-subsets of \([m + n]\) and let \(S_r = \{ X \in S : |X \cap A| = r \}\). Then

- \(S_0, S_1, \ldots, S_k\) is a partition of \(S\).
- \(|S_0| + |S_1| + \cdots + |S_k| = |S|\).
- \(|S_r| = \binom{m}{r} \binom{n}{k-r}\).
- \(|S| = \binom{m+n}{k}\).
**Theorem**

*Binomial Theorem*

\[(1 + x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.\]

**Proof**  
Coefficient \(x^r\) in \((1 + x)(1 + x) \cdots (1 + x)\): choose \(x\) from \(r\) brackets and 1 from the rest. 

□
Applications of Binomial Theorem

- \( x = 1: \)

\[
\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = (1 + 1)^n = 2^n.
\]

LHS counts the number of subsets of all sizes in \([n]\).

- \( x = -1: \)

\[
\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = (1 - 1)^n = 0,
\]

i.e.

\[
\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots
\]

and number of subsets of even cardinality = number of subsets of odd cardinality.
\[ \sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}. \]

Differentiate both sides of the Binomial Theorem w.r.t. \( x \).

\[ n(1 + x)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} x^{k-1}. \]

Now put \( x = 1 \).
Inclusion-Exclusion

2 sets:

\[ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \]

So if \( A_1, A_2 \subseteq A \) and \( \bar{A}_i = A \setminus A_i, i = 1, 2 \) then

\[ |\bar{A}_1 \cap \bar{A}_2| = |A| - |A_1| - |A_2| + |A_1 \cap A_2| \]

3 sets:

\[ |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |A| - |A_1| - |A_2| - |A_3| \]
\[ + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \]
\[ - |A_1 \cap A_2 \cap A_3|. \]
General Case

\( A_1, A_2, \ldots, A_N \subseteq A \) and each \( x \in A \) has a weight \( w_x \). (In our examples \( w_x = 1 \) for all \( x \) and so \( w(X) = |X| \).)

For \( S \subseteq [N] \), \( A_S = \bigcap_{i \in S} A_i \) and \( w(S) = \sum_{x \in S} w_x \).

E.g. \( A_{\{4,7,18\}} = A_4 \cap A_7 \cap A_{18} \).

\( A_\emptyset = A \).

Inclusion-Exclusion Formula:

\[
w \left( \bigcap_{i=1}^{N} A_i \right) = \sum_{S \subseteq [N]} (-1)^{|S|} w(A_S) .
\]
Simple example. How many integers in \([1000]\) are not divisible by 5, 6 or 8 i.e. what is the size of \(A_1 \cap A_2 \cap A_3\) below? Here we take \(w_x = 1\) for all \(x\).

\[
\begin{align*}
A &= A_0 &= \{1, 2, 3, \ldots, \} & |A| = 1000 \\
A_1 &= \{5, 10, 15, \ldots, \} & |A_1| = 200 \\
A_2 &= \{6, 12, 18, \ldots, \} & |A_2| = 166 \\
A_3 &= \{8, 16, 24, \ldots, \} & |A_3| = 125 \\
A_{\{1,2\}} &= \{30, 60, 90, \ldots, \} & |A_{\{1,2\}}| = 33 \\
A_{\{1,3\}} &= \{40, 80, 120, \ldots, \} & |A_{\{1,3\}}| = 25 \\
A_{\{2,3\}} &= \{24, 48, 72, \ldots, \} & |A_{\{2,3\}}| = 41 \\
A_{\{1,2,3\}} &= \{120, 240, 360, \ldots, \} & |A_{\{1,2,3\}}| = 8
\end{align*}
\]

\[
|A_1 \cap A_2 \cap A_3| = 1000 - (200 + 166 + 125) + (33 + 25 + 41) - 8 = 600.
\]
A derangement of $[n]$ is a permutation $\pi$ such that

$$\pi(i) \neq i : \quad i = 1, 2, \ldots, n.$$ 

We must express the set of derangements $D_n$ of $[n]$ as the intersection of the complements of sets.

We let $A_i = \{\text{permutations } \pi : \pi(i) = i\}$ and then

$$|D_n| = \left| \bigcap_{i=1}^{n} \overline{A}_i \right| .$$
We must now compute $|A_S|$ for $S \subseteq [n]$.

$|A_1| = (n - 1)!$: after fixing $\pi(1) = 1$ there are $(n - 1)!$ ways of permuting $2, 3, \ldots, n$.

$|A_{\{1,2\}}| = (n - 2)!$: after fixing $\pi(1) = 1, \pi(2) = 2$ there are $(n - 2)!$ ways of permuting $3, 4, \ldots, n$.

In general

$|A_S| = (n - |S|)!$
\begin{align*}
|D_n| &= \sum_{S \subseteq [n]} (-1)^{|S|}(n - |S|)!
\quad = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k}(n - k)!
\quad = \sum_{k=0}^{n} (-1)^{k} \frac{n!}{k!}
\quad = n! \sum_{k=0}^{n} (-1)^{k} \frac{1}{k!}.
\end{align*}

When \( n \) is large,

\[
\sum_{k=0}^{n} (-1)^{k} \frac{1}{k!} \approx e^{-1}.
\]
Proof of inclusion-exclusion formula

\[ \theta_{x,i} = \begin{cases} 
1 & x \in A_i \\
0 & x \notin A_i 
\end{cases} \]

\[ (1 - \theta_{x,1})(1 - \theta_{x,2}) \cdots (1 - \theta_{x,N}) = \begin{cases} 
1 & x \in \bigcap_{i=1}^{N} \overline{A_i} \\
0 & \text{otherwise} \end{cases} \]

So

\[ w \left( \bigcap_{i=1}^{N} \overline{A_i} \right) = \sum_{x \in A} w_x (1 - \theta_{x,1})(1 - \theta_{x,2}) \cdots (1 - \theta_{x,N}) \]

\[ = \sum_{x \in A} w_x \sum_{S \subseteq [N]} (-1)^{|S|} \prod_{i \in S} \theta_{x,i} \]

\[ = \sum_{S \subseteq [N]} (-1)^{|S|} w(A_S) \]
Euler’s Function $\phi(n)$.

Let $\phi(n)$ be the number of positive integers $x \leq n$ which are mutually prime to $n$ i.e. have no common factors with $n$, other than 1. 

$\phi(12) = 4$.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_1^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorisation of $n$.

$$A_i = \{ x \in [n] : p_i \text{ divides } x \}, \quad 1 \leq i \leq k.$$ 

$$\phi(n) = \left| \bigcap_{i=1}^{k} \overline{A_i} \right|$$
\[ |A_S| = \frac{n}{\prod_{i \in S} p_i} \quad S \subseteq [k]. \]

\[
\phi(n) = \sum_{S \subseteq [k]} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i}
\]

\[
= n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)
\]
Surjections

Fix $n, m$. Let

$$A = \{ f : [n] \rightarrow [m] \}$$

Thus $|A| = m^n$. Let

$$F(n, m) = \{ f \in A : f \text{ is onto } [m] \}.$$ 

How big is $F(n, m)$?

Let

$$A_i = \{ f \in F : f(x) \neq i, \forall x \in [n] \}.$$ 

Then

$$F(n, m) = \bigcap_{i=1}^{m} \overline{A}_i.$$
For $S \subseteq [m]$

$$A_S = \{ f \in A : f(x) \not\in S, \forall x \in [n] \}.$$  
$$= \{ f : [n] \rightarrow [m] \setminus S \}.$$  

So

$$|A_S| = (m - |S|)^n.$$  

Hence

$$F(n, m) = \sum_{S \subseteq [m]} (-1)^{|S|} (m - |S|)^n$$  

$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m - k)^n.$$
Scrambled Allocations

We have $n$ boxes $B_1, B_2, \ldots, B_n$ and $2n$ distinguishable balls $b_1, b_2, \ldots, b_{2n}$. An allocation of balls to boxes, two balls to a box, is said to be scrambled if there does not exist $i$ such that box $B_i$ contains balls $b_{2i-1}, b_{2i}$. Let $\sigma_n$ be the number of scrambled allocations.

Let $A_i$ be the set of allocations in which box $B_i$ contains $b_{2i-1}, b_{2i}$. We show that

$$|A_S| = \frac{(2(n - |S|))!}{2^{n-|S|}}.$$  

Inclusion-Exclusion then gives

$$\sigma_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2(n - k))!}{2^{n-k}}.$$  

Covered so far
First consider $A_\emptyset$:

Each permutation $\pi$ of $[2n]$ yields an allocation of balls, placing $b_{\pi(2i-1)}, b_{\pi(2i)}$ into box $B_i$, for $i = 1, 2, \ldots, n$. The order of balls in the boxes is immaterial and so each allocation comes from exactly $2^n$ distinct permutations, giving

$$|A_\emptyset| = \frac{(2n)!}{2^n}.$$

To get the formula for $|A_S|$ observe that the contents of $2|S|$ boxes are fixed and so we are in essence dealing with $n - |S|$ boxes and $2(n - |S|)$ balls.
Problème des Ménages

In how many ways $M_n$ can $n$ male-female couples be seated around a table, alternating male-female, so that no person is seated next to their partner?

Let $A_i$ be the set of seatings in which couple $i$ sit together.

If $|S| = k$ then

\[ |A_S| = 2k!(n - k)!^2 \times d_k. \]

$d_k$ is the number of ways of placing $k$ 1’s on a cycle of length $2n$ so that no two 1’s are adjacent. (We place a person at each 1 and his/her partner on the succeeding 0).

2 choices for which seats are occupied by the men or women. $k!$ ways of assigning the couples to the positions; $(n - k)!^2$ ways of assigning the rest of the people.
\[ d_k = \frac{2n}{k} \binom{2n - k - 1}{k - 1} = \frac{2n}{2n - k} \binom{2n - k}{k}. \]

(See slides 11 and 12).

\[ M_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \times 2k!(n - k)!^2 \times \frac{2n}{2n - k} \binom{2n - k}{k} \]

\[ = 2n! \sum_{k=0}^{n} (-1)^k \frac{2n}{2n - k} \binom{2n - k}{k} (n - k)!. \]
The weight of elements in exactly \( k \) sets:

Observe that

\[
\prod_{i \in S} \theta_{x,i} \prod_{i \notin S} (1 - \theta_{x,i}) = 1 \iff x \in A_i, i \in S \text{ and } x \notin A_i, i \notin S.
\]

\( W_k \) is the total weight of elements in exactly \( k \) of the \( A_i \):

\[
N_k = \sum_{x \in A} \sum_{\substack{|S| = k \quad x \in A \quad i \in S \quad i \notin S}} \prod_{i \in S} \theta_{x,i} \prod_{i \notin S} (1 - \theta_{x,i})
= \sum_{|S| = k} \sum_{x \in A} \sum_{i \in S} \prod_{i \notin S} \theta_{x,i} \prod_{i \notin S} (1 - \theta_{x,i})
= \sum_{|S| = k} \sum_{T \supseteq S} \sum_{x \in A} (-1)^{|T \setminus S|} \prod_{i \in T} \theta_{x,i}
= \sum_{|S| = k} \sum_{T \supseteq S} (-1)^{|T \setminus S|} w(A_T)
= \sum_{\ell = k}^N \sum_{|T| = \ell} (-1)^{\ell - k} \binom{\ell}{k} w(A_T).
\]

Covered so far
As an example. Let \( D_{n,k} \) denote the number of permutations \( \pi \) of \( [n] \) for which there are exactly \( k \) indices \( i \) for which \( \pi(i) = i \). Then

\[
D_{n,k} = \sum_{\ell=k}^{n} \binom{n}{\ell} (-1)^{\ell-k} \binom{\ell}{k} (n-\ell)!
\]

\[
= \sum_{\ell=k}^{n} \frac{n!}{\ell!(n-\ell)!} (-1)^{\ell-k} \frac{\ell!}{k!(\ell-k)!} (n-\ell)!
\]

\[
= \frac{n!}{k!} \sum_{\ell=k}^{n} \frac{(-1)^{\ell-k}}{(\ell-k)!}
\]

\[
= \frac{n!}{k!} \sum_{r=0}^{n-k} \frac{(-1)^{r}}{r!}
\]

\[
\approx \frac{n!}{ek!}
\]

when \( n \) is large and \( k \) is constant.
9/8/2021
Recurrence Relations

Suppose $a_0, a_1, a_2, \ldots, a_n, \ldots$, is an infinite sequence. A recurrence relation is a set of equations

$$a_n = f_n(a_{n-1}, a_{n-2}, \ldots, a_{n-k}). \quad (3)$$

The whole sequence is determined by (18) and the values of $a_0, a_1, \ldots, a_{k-1}$.
Linear Recurrence

Fibonacci Sequence

\[ a_n = a_{n-1} + a_{n-2} \quad n \geq 2. \]

\[ a_0 = a_1 = 1. \]
\[ b_n = |B_n| = |\{ x \in \{a, b, c\}^n : aa \text{ does not occur in } x \}|. \]

\[ b_1 = 3 : a \ b \ c \]

\[ b_2 = 8 : \ ab \ ac \ ba \ bb \ bc \ ca \ cb \ cc \]

\[ b_n = 2b_{n-1} + 2b_{n-2} \quad n \geq 2. \]
\[ b_n = 2b_{n-1} + 2b_{n-2} \quad n \geq 2. \]

Let

\[ B_n = B_n^{(b)} \cup B_n^{(c)} \cup B_n^{(a)} \]

where \( B_n^{(\alpha)} = \{ x \in B_n : x_1 = \alpha \} \) for \( \alpha = a, b, c \).

Now \( |B_n^{(b)}| = |B_n^{(c)}| = |B_{n-1}| \). The map \( f : B_n^{(b)} \to B_{n-1} \),

\[ f(bx_2x_3 \ldots x_n) = x_2x_3 \ldots x_n \]

is a bijection.

\[ B_n^{(a)} = \{ x \in B_n : x_1 = a \text{ and } x_2 = b \text{ or } c \} \]. The map \( g : B_n^{(a)} \to B_{n-1}^{(b)} \cup B_{n-1}^{(c)} \),

\[ g(ax_2x_3 \ldots x_n) = x_2x_3 \ldots x_n \]

is a bijection.

Hence, \( |B_n^{(a)}| = 2|B_{n-2}| \).
Towers of Hanoi

$H_n$ is the minimum number of moves needed to shift $n$ rings from Peg 1 to Peg 2. One is not allowed to place a larger ring on top of a smaller ring.
We see that $H_1 = 1$ and $H_n = 2H_{n-1} + 1$ for $n \geq 2$.

So,

$$\frac{H_n}{2^n} - \frac{H_{n-1}}{2^{n-1}} = \frac{1}{2^n}.$$

Summing these equations give

$$\frac{H_n}{2^n} - \frac{H_1}{2} = \frac{1}{2^n} + \frac{1}{2^{n-1}} + \cdots + \frac{1}{4} = \frac{1}{2} - \frac{1}{2^n}.$$

So

$$H_n = 2^n - 1.$$
$A$ has $n$ dollars. Everyday $A$ buys one of a Bun (1 dollar), an Ice-Cream (2 dollars) or a Pastry (2 dollars). How many ways are there (sequences) for $A$ to spend his money? Ex. $BBPIIPBI$ represents “Day 1, buy Bun. Day 2, buy Bun etc.”.

$$u_n = \text{number of ways} = u_{n,B} + u_{n,I} + u_{n,P}$$

where $u_{n,B}$ is the number of ways where $A$ buys a Bun on day 1 etc. $u_{n,B} = u_{n-1}$, $u_{n,I} = u_{n,P} = u_{n-2}$. So

$$u_n = u_{n-1} + 2u_{n-2},$$

and

$$u_0 = u_1 = 1.$$
If $a_0, a_1, \ldots, a_n$ is a sequence of real numbers then its (ordinary) generating function $a(x)$ is given by

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + \cdots$$

and we write

$$a_n = [x^n]a(x).$$

For more on this subject see Generatingfunctionology by the late Herbert S. Wilf. The book is available from https://www.math.upenn.edu// wilf/DownIdGF.html
\[ a_n = 1 \]

\[ a(x) = \frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^n + \cdots \]

\[ a_n = n + 1. \]

\[ a(x) = \frac{1}{(1 - x)^2} = 1 + 2x + 3x^2 + \cdots + (n + 1)x^n + \cdots \]

\[ a_n = n. \]

\[ a(x) = \frac{x}{(1 - x)^2} = x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots \]
Generalised binomial theorem:

\[ a_n = \binom{\alpha}{n} \]

\[ a(x) = (1 + x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n. \]

where

\[ \binom{\alpha}{n} = \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - n + 1)}{n!} \]

\[ a_n = \binom{m+n-1}{n} \]

\[ a(x) = \frac{1}{(1 - x)^m} = \sum_{n=0}^{\infty} \binom{-m}{n} (-x)^n = \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n. \]
General view.

Given a recurrence relation for the sequence \((a_n)\), we

(a) Deduce from it, an equation satisfied by the generating function \(a(x) = \sum_n a_n x^n\).

(b) Solve this equation to get an explicit expression for the generating function.

(c) Extract the coefficient \(a_n\) of \(x^n\) from \(a(x)\), by expanding \(a(x)\) as a power series.
Solution of linear recurrences

\[ a_n - 6a_{n-1} + 9a_{n-2} = 0 \quad n \geq 2. \]

\[ a_0 = 1, \ a_1 = 9. \]

\[ \sum_{n=2}^{\infty} (a_n - 6a_{n-1} + 9a_{n-2})x^n = 0. \quad (4) \]
\[ \sum_{n=2}^{\infty} a_n x^n = a(x) - a_0 - a_1 x \]
\[ = a(x) - 1 - 9x. \]
\[ \sum_{n=2}^{\infty} 6a_{n-1} x^n = 6x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} \]
\[ = 6x(a(x) - a_0) \]
\[ = 6x(a(x) - 1). \]
\[ \sum_{n=2}^{\infty} 9a_{n-2} x^n = 9x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \]
\[ = 9x^2 a(x). \]
\[ a(x) - 1 - 9x - 6x(a(x) - 1) + 9x^2 a(x) = 0 \]

or

\[ a(x)(1 - 6x + 9x^2) - (1 + 3x) = 0. \]

\[ a(x) = \frac{1 + 3x}{1 - 6x + 9x^2} = \frac{1 + 3x}{(1 - 3x)^2} \]

\[ = \sum_{n=0}^{\infty} (n + 1)3^n x^n + 3x \sum_{n=0}^{\infty} (n + 1)3^n x^n \]

\[ = \sum_{n=0}^{\infty} (n + 1)3^n x^n + \sum_{n=0}^{\infty} n3^n x^n \]

\[ = \sum_{n=0}^{\infty} (2n + 1)3^n x^n. \]

\[ a_n = (2n + 1)3^n. \]
Fibonacci sequence:

\[
\sum_{n=2}^{\infty} (a_n - a_{n-1} - a_{n-2}) x^n = 0.
\]

\[
\sum_{n=2}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0.
\]

\[
(a(x) - a_0 - a_1 x) - (x(a(x) - a_0)) - x^2 a(x) = 0.
\]

\[
a(x) = \frac{1}{1 - x - x^2}.
\]
\[ a(x) = -\frac{1}{(\xi_1 - x)(\xi_2 - x)} \]

\[ = \frac{1}{\xi_1 - \xi_2} \left( \frac{1}{\xi_1 - x} - \frac{1}{\xi_2 - x} \right) \]

\[ = \frac{1}{\xi_1 - \xi_2} \left( \frac{\xi_1^{-1}}{1 - x/\xi_1} - \frac{\xi_2^{-1}}{1 - x/\xi_2} \right) \]

where

\[ \xi_1 = -\frac{\sqrt{5} + 1}{2} \text{ and } \xi_2 = \frac{\sqrt{5} - 1}{2} \]

are the 2 roots of

\[ x^2 + x - 1 = 0. \]
Therefore,

\[
\begin{align*}
    a(x) &= \frac{\xi_1^{-1}}{\xi_1 - \xi_2} \sum_{n=0}^{\infty} \xi_1^{-n} x^n - \frac{\xi_2^{-1}}{\xi_1 - \xi_2} \sum_{n=0}^{\infty} \xi_2^{-n} x^n \\
    &= \sum_{n=0}^{\infty} \frac{\xi_1^{-n-1} - \xi_2^{-n-1}}{\xi_1 - \xi_2} x^n
\end{align*}
\]

and so

\[
\begin{align*}
    a_n &= \frac{\xi_1^{-n-1} - \xi_2^{-n-1}}{\xi_1 - \xi_2} \\
    &= \frac{1}{\sqrt{5}} \left( \left( \frac{\sqrt{5} + 1}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right).
\end{align*}
\]
Inhomogeneous problem

\[ a_n - 3a_{n-1} = n^2 \quad n \geq 1. \]

\[ a_0 = 1. \]

\[
\sum_{n=1}^{\infty} (a_n - 3a_{n-1})x^n = \sum_{n=1}^{\infty} n^2 x^n
\]

\[
\sum_{n=1}^{\infty} n^2 x^n = \sum_{n=2}^{\infty} n(n-1)x^n + \sum_{n=1}^{\infty} nx^n
\]

\[
= \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}
\]

\[
= \frac{x + x^2}{(1-x)^3}
\]

\[
\sum_{n=1}^{\infty} (a_n - 3a_{n-1})x^n = a(x) - 1 - 3xa(x)
\]

\[ = a(x)(1 - 3x) - 1. \]
\[
a(x) = \frac{x + x^2}{(1 - x)^3(1 - 3x)} + \frac{1}{1 - 3x}
\]
\[
= \frac{A}{1 - x} + \frac{B}{(1 - x)^2} + \frac{C}{(1 - x)^3} + \frac{D + 1}{1 - 3x}
\]

where
\[
x + x^2 = A(1 - x)^2(1 - 3x) + B(1 - x)(1 - 3x) + C(1 - 3x) + D(1 - x)^3.
\]

Then
\[
A = -1/2, \quad B = 0, \quad C = -1, \quad D = 3/2.
\]
So

\[ a(x) = \frac{-1/2}{1-x} - \frac{1}{(1-x)^3} + \frac{5/2}{1-3x} \]

\[ = -\frac{1}{2} \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} \binom{n+2}{2} x^n + \frac{5}{2} \sum_{n=0}^{\infty} 3^n x^n \]

So

\[ a_n = -\frac{1}{2} - \binom{n+2}{2} + \frac{5}{2} 3^n \]

\[ = -\frac{3}{2} - \frac{3n}{2} - \frac{n^2}{2} + \frac{5}{2} 3^n. \]
Products of generating functions

\[ a(x) = \sum_{n=0}^{\infty} a_n x^n, \quad b(x) = \sum_{n=0}^{\infty} b_n x^n. \]

\[ a(x)b(x) = (a_0 + a_1 x + a_2 x^2 + \cdots) \times (b_0 + b_1 x + b_2 x^2 + \cdots) \]

\[ = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots \]

\[ = \sum_{n=0}^{\infty} c_n x^n \]

where

\[ c_n = \sum_{k=0}^{n} a_k b_{n-k}. \]
Derangements

\[ n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}. \]

Explanation: \( \binom{n}{k} d_{n-k} \) is the number of permutations with exactly \( k \) cycles of length 1. Choose \( k \) elements \( \binom{n}{k} \) ways for which \( \pi(i) = i \) and then choose a derangement of the remaining \( n - k \) elements.

So

\[ 1 = \sum_{k=0}^{n} \frac{1}{k! (n-k)!} d_{n-k} \]

\[ \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{1}{k! (n-k)!} d_{n-k} \right) x^n. \]
Let 
\[ d(x) = \sum_{m=0}^{\infty} \frac{d_m}{m!} x^m. \]

From (5) we have
\[ \frac{1}{1 - x} = e^x d(x) \]
\[ d(x) = \frac{e^{-x}}{1 - x} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \frac{(-1)^k}{k!} \right) x^n. \]

So
\[ \frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \]
Triangulation of $n$-gon

Let

$$a_n = \text{number of triangulations of } P_{n+1}$$

$$= \sum_{k=0}^{n} a_k a_{n-k} \quad n \geq 2$$

$$a_0 = 0, a_1 = a_2 = 1.$$
Explanation of (6):

\(a_k a_{n-k}\) counts the number of triangulations in which edge 1, \(n+1\) is contained in triangle 1, \(k+1\), \(n+1\).

There are \(a_k\) ways of triangulating 1, 2, \ldots, \(k+1\), 1 and for each such there are \(a_{n-k}\) ways of triangulating \(k+1\), \(k+2\), \ldots, \(n+1\), \(k+1\).
\[ x + \sum_{n=2}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} \left( \sum_{k=0}^{n} a_k a_{n-k} \right) x^n. \]

But,

\[ x + \sum_{n=2}^{\infty} a_n x^n = a(x) \]

since \( a_0 = 0, a_1 = 1. \)

\[ \sum_{n=2}^{\infty} \left( \sum_{k=0}^{n} a_k a_{n-k} \right) x^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k a_{n-k} \right) x^n = a(x)^2. \]
So

\[ a(x) = x + a(x)^2 \]

and hence

\[ a(x) = \frac{1 + \sqrt{1 - 4x}}{2} \quad \text{or} \quad \frac{1 - \sqrt{1 - 4x}}{2}. \]

But \( a(0) = 0 \) and so

\[ a(x) = \frac{1 - \sqrt{1 - 4x}}{2} = \frac{1}{2} - \frac{1}{2} \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^{2n-1}} \binom{2n-2}{n-1} (-4x)^n \right) \]

\[ = \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n. \]

So

\[ a_n = \frac{1}{n} \binom{2n-2}{n-1}. \]
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Theorem

Let $A_1, A_2, \ldots, A_n$ be subsets of $A$ and $|A_i| = k$ for $1 \leq i \leq n$. If $n < 2^{k-1}$ then there exists a partition $A = R \cup B$ such that

$A_i \cap R \neq \emptyset$ and $A_i \cap B \neq \emptyset \quad 1 \leq i \leq n.$

[$R = \text{Red elements and } B = \text{Blue elements}.$]

Proof

Randomly colour $A$.

$\Omega = \{R, B\}^A = \{f : A \to \{R, B\}\}$, uniform distribution.

$BAD = \{\exists i : A_i \subseteq R \text{ or } A_i \subseteq B\}.$

Claim: $\Pr(BAD) < 1$.

Thus $\Omega \setminus BAD \neq \emptyset$ and this proves the theorem.
\[ \text{BAD}(i) = \{ A_i \subseteq R \text{ or } A_i \subseteq B \} \text{ and } \text{BAD} = \bigcup_{i=1}^{n} \text{BAD}(i). \]

**Boole's Inequality:** if \( \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N \) are a collection of events, then

\[
\Pr \left( \bigcup_{i=1}^{N} \mathcal{A}_i \right) \leq \sum_{i=1}^{N} \Pr(\mathcal{A}_i).
\]

This easily proved by induction on \( N \). When \( N = 2 \) we use

\[
\Pr(\mathcal{A}_1 \cup \mathcal{A}_2) = \Pr(\mathcal{A}_1) + \Pr(\mathcal{A}_2) - \Pr(\mathcal{A}_1 \cap \mathcal{A}_2) \leq \Pr(\mathcal{A}_1 \cup \mathcal{A}_2).
\]

In general,

\[
\Pr \left( \bigcup_{i=1}^{N} \mathcal{A}_i \right) \leq \Pr \left( \bigcup_{i=1}^{N-1} \mathcal{A}_i \right) + \Pr(\mathcal{A}_N) \leq \sum_{i=1}^{N-1} \Pr(\mathcal{A}_i) + \Pr(\mathcal{A}_N).
\]

The first inequality is the two event case and the second is by induction on \( N \).
So,

\[
\Pr(BAD) \leq \sum_{i=1}^{n} \Pr(BAD(i))
\]

\[
= \sum_{i=1}^{n} \left( \frac{1}{2} \right)^{k-1}
\]

\[
= \frac{n}{2^{k-1}}
\]

\[
< 1.
\]
Example of system which is not 2-colorable.

Let \( n = \binom{2k-1}{k} \) and \( A = [2k - 1] \) and

\[
\{A_1, A_2, \ldots, A_n\} = \binom{[2k-1]}{k}.
\]

Then in any 2-coloring of \( A_1, A_2, \ldots, A_n \) there is a set \( A_i \) all of whose elements are of one color.

Suppose \( A \) is partitioned into 2 sets \( R, B \). At least one of these two sets is of size at least \( k \) (since \((k - 1) + (k - 1) < 2k - 1\)).

Suppose then that \( R \geq k \) and let \( S \) be any \( k \)-subset of \( R \). Then there exists \( i \) such that \( A_i = S \subseteq R \).
Tournaments

$n$ players in a tournament each play each other i.e. there are \( \binom{n}{2} \) games.

Fix some $k$. Is it possible that for every set $S$ of $k$ players there is a person $w_S$ who beats everyone in $S$?
Suppose that the results of the tournament are decided by a random coin toss.

Fix \( S, \ |S| = k \) and let \( \mathcal{E}_S \) be the event that nobody beats everyone in \( S \).

The event

\[
\mathcal{E} = \bigcup_S \mathcal{E}_S
\]

is that there is a set \( S \) for which \( w_S \) does not exist.

We only have to show that \( \Pr(\mathcal{E}) < 1 \).
\[ \Pr(\mathcal{E}) \leq \sum_{|S|=k} \Pr(\mathcal{E}_S) \]
\[ = \binom{n}{k} (1 - 2^{-k})^{n-k} \]
\[ < n^k e^{-(n-k)2^{-k}} \]
\[ = \exp\{ k \ln n - (n - k)2^{-k} \} \]
\[ \rightarrow 0 \]

since we are assuming here that \( k \) is fixed independent of \( n \).
Graph Crossing Number

The crossing number of a graph $G$ is the minimum number of edge crossings of a drawing of $G$ in the plane.

Euler’s formula implies that a planar graph with $n$ vertices has at most $3n$ edges.

This implies that a graph $G = (V, E)$ requires at least $|E| - 3|V|$ crossings.

**Theorem**

*If $|E| > 4|V|$ then $G$ has crossing number $\Omega(|E|^3/|V|^2)$.***

If $|E| \approx |V|^{3/2}$ then this gives $\Omega(|V|^{5/2})$ whereas $|E| - 3|V| = O(|V|^{3/2})$. 

Covered so far
Suppose that $G$ has a drawing with $k$ crossings and let $0 < p < 1$.

Let $G_p = (V_p, E_p)$ denote the subgraph of $G$ obtained by including each vertex in $V_p$ independently with probability $p$.

$E_p$ is then the set of edges $\{x, y\}$ such that $x, y \in V_p$.

$$E(|V_p|) = p|V| \text{ and } E(|E_p| = p^2|E|).$$

Also,

$$E(\text{number of crossings in the drawing of } G_p) = p^4k.$$
So,

\[ p^4 k \geq \mathbb{E}(|E_p| - 3|V_p|) = p^2|E| - 3p|V|. \]

So

\[ k \geq \frac{p^2|E| - 3p|V|}{p^4}. \]

Maximising the RHS over \( p \leq 1 \) gives \( p = 4|V|/|E| \) and

\[ k \geq \frac{|E|^3}{64|V|^2}. \]
A binary tree consists of a set of nodes, one of which is the root. Each node is connected to 0, 1 or 2 nodes below it and every node other than the root is connected to exactly one node above it. The root is the highest node. The depth of a node is the number of edges in its path to the root. The depth of a tree is the maximum over the depths of its nodes.
Starting with a tree $T_0$ consisting of a single root $r$, we grow a tree $T_n$ as follows:

The $n$'th particle starts at $r$ and flips a fair coin. It goes left (L) with probability 1/2 and right (R) with probability 1/2.

It tries to move along the tree in the chosen direction. If there is a node below it in this direction then it goes there and continues its random moves. Otherwise it creates a new node where it wanted to move and stops.
Let $D_n$ be the depth of this tree. 

**Claim:** for any $t \geq 0$,

$$\Pr(D_n \geq t) \leq (n2^{-(t-1)/2})^t.$$

**Proof** The process requires at most $n^2$ coin flips and so we let $\Omega = \{L, R\}^{n^2}$ – most coin flips will not be needed most of the time.

$$DEEP = \{D_n \geq t\}.$$

For $P \in \{L, R\}^t$ and $S \subseteq [n]$, $|S| = t$ let $DEEP(P, S) = \{\text{the particles } S = \{s_1, s_2, \ldots, s_t\} \text{ follow } P \text{ in the tree i.e. the first } i \text{ moves of } s_i \text{ are along } P, 1 \leq i \leq t\}.

$$DEEP = \bigcup_P \bigcup_S DEEP(P, S).$$
t=5 and DEEP(P,S) occurs if
4 goes L...
8 goes LR...
17 goes LRR...
11 goes LRRL...
13 goes LRRLR...

S={4,8,11,13,17}
\[ \Pr(\text{DEEP}) \leq \sum_{P} \sum_{S} \Pr(\text{DEEP}(P, S)) \]
\[ = \sum_{P} \sum_{S} 2^{-(1+2+\cdots+t)} \]
\[ = \sum_{P} \sum_{S} 2^{-t(t+1)/2} \]
\[ = 2^t \binom{n}{t} 2^{-t(t+1)/2} \]
\[ \leq 2^t n^t 2^{-t(t+1)/2} \]
\[ = (n2^{-(t-1)/2})^t. \]

So if we put \( t = A \log_2 n \) then

\[ \Pr(D_n \geq A \log_2 n) \leq (2n^{1-A/2})^{A \log_2 n} \]

which is very small, for \( A > 2 \).
A problem with hats

There are $n$ people standing a circle. They are blind-folded and someone places a hat on each person’s head. The hat has been randomly colored Red or Blue.

They take off their blind-folds and everyone can see everyone else’s hat. Each person then simultaneously declares (i) my hat is red or (ii) my hat is blue or (iii) or I pass.

They win a big prize if the people who opt for (i) or (ii) are all correct. They pay a big penalty if there is a person who incorrectly guesses the color of their hat.

Is there a strategy which means they will win with probability better than $1/2$?
Suppose that we partition $Q_n = \{0, 1\}^n$ into 2 sets $W, L$ which have the property that $L$ is a cover i.e. if $x = x_1x_2 \cdots x_n \in W = Q_n \setminus L$ then there is $y_1y_2 \cdots y_n \in L$ such that $h(x, y) = 1$ where

$$h(x, y) = |\{j : x_j \neq y_j\}|.$$

Hamming distance between $x$ and $y$.

Assume that $0 \equiv \text{Red}$ and $1 \equiv \text{Blue}$. Person $i$ knows $x_j$ for $j \neq i$ (color of hat $j$) and if there is a unique value $\xi$ of $x_i$ which places $x$ in $W$ then person $i$ will declare that their hat has color $\xi$.

The people assume that $x \in W$ and if indeed $x \in W$ then there is at least one person who will be in this situation and any such person will guess correctly.

Is there a small cover $L$?
Let \( p = \frac{\ln n}{n} \). Choose \( L_1 \) randomly by placing \( y \in Q_n \) into \( L_1 \) with probability \( p \).

Then let \( L_2 \) be those \( z \in Q_n \) which are not at Hamming distance \( \leq 1 \) from some member of \( L_1 \).

Clearly \( L = L_1 \cup L_2 \) is a cover and
\[
E(|L|) = 2^n p + 2^n (1 - p)^{n+1} \leq 2^n (p + e^{-np}) \leq 2^{n \frac{2 \ln n}{n}}.
\]

So there must exist a cover of size at most \( 2^{n \frac{2 \ln n}{n}} \) and the players can win with probability at least \( 1 - \frac{2 \ln n}{n} \).
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Hoeffding’s Inequality – I
Let $X_1, X_2, \ldots, X_n$ be independent random variables taking values such that $\Pr(X_i = 1) = 1/2 = \Pr(X_i = -1)$ for $i = 1, 2, \ldots, n$. Let $X = X_1 + X_2 + \cdots + X_n$. Then for any $t \geq 0$

$$\Pr(|X| \geq t) < 2e^{-t^2/2n}.$$  

Proof: For any $\lambda > 0$ we have

$$\Pr(X \geq t) = \Pr(e^{\lambda X} \geq e^{\lambda t}) \leq e^{-\lambda t}E(e^{\lambda X}).$$

Now for $i = 1, 2, \ldots, n$ we have

$$E(e^{\lambda X_i}) = \frac{e^{-\lambda} + e^{\lambda}}{2} = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots < e^{\lambda^2/2}.$$
So, by independence,

\[ E(e^{\lambda X}) = E \left( \prod_{i=1}^{n} e^{\lambda X_i} \right) = \prod_{i=1}^{n} E(e^{\lambda X_i}) \leq e^{\lambda^2 n/2}. \]

Hence,

\[ \Pr(X \geq t) \leq e^{-\lambda t + \lambda^2 n/2}. \]

We choose \( \lambda = t/n \) to minimise \( -\lambda t + \lambda^2 n/2 \). This yields

\[ \Pr(X \geq t) \leq e^{-t^2/2n}. \]

Similarly,

\[ \Pr(X \leq -t) = \Pr(e^{-\lambda X} \geq e^{\lambda t}) \leq e^{-\lambda t} E(e^{-\lambda X}) \leq e^{-\lambda t + \lambda^2 n/2}. \]
Suppose that $|X| = n$ and $\mathcal{F} \subseteq \mathcal{P}(X)$. If we color the elements of $X$ with Red and Blue i.e. partition $X$ in $R \cup B$ then the discrepancy $\text{disc}(\mathcal{F}, R, B)$ of this coloring is defined

$$\text{disc}(\mathcal{F}, R, B) = \max_{F \in \mathcal{F}} \text{disc}(F, R, B)$$

where $\text{disc}(F, R, B) = |R \cap F| - |B \cap F|$ i.e. the absolute difference between the number of elements of $F$ that are colored Red and the number that are colored Blue.
Claim:

If $|\mathcal{F}| = m$ then there exists a coloring $R, B$ such that $disc(\mathcal{F}, R, B) \leq (2n \log_e(2m))^{1/2}$.

**Proof** Fix $F \in \mathcal{F}$ and let $s = |F|$. If we color $X$ randomly and let $Z = |R \cap F| - |B \cap F|$ then $Z$ is the sum of $s$ independent $\pm 1$ random variables.

So, by the Hoeffding inequality,

$$\Pr(|Z| \geq (2n \log_e(2m))^{1/2}) < 2e^{-n \log_e(2m)/s} \leq \frac{1}{m}.$$
The Local Lemma

We go back to the coloring problem at the beginning of these slides. We now place a different restriction on the sets involved.

**Theorem**

Let $A_1, A_2, \ldots, A_n$ be subsets of $A$ where $|A_i| \geq k$ for $1 \leq i \leq n$. If each $A_i$ intersects at most $2^k - 3$ other sets then there exists a partition $A = R \cup B$ such that

$$A_i \cap R \neq \emptyset \text{ and } A_i \cap B \neq \emptyset \quad 1 \leq i \leq n.$$
Symmetric Local Lemma: We consider the following situation. 
\( X = \{x_1, x_2, \ldots, x_N\} \) is a collection of independent random variables. Suppose that we have events \( \mathcal{E}_i, i = 1, 2, \ldots, m \) where \( \mathcal{E}_i \) depends only on the set \( X_i \subseteq X \). Thus if \( X_i \cap X_j = \emptyset \) then \( \mathcal{E}_i \) and \( \mathcal{E}_j \) are independent. 
The dependency graph \( \Gamma \) has vertex set \([m]\) and an edge \((i, j)\) iff \( X_i \cap X_j \neq \emptyset \).

**Theorem**

Let

\[
p = \max_i \Pr(\mathcal{E}_i)\text{ and let } d \text{ be the maximum degree of } \Gamma.
\]

\[
4dp \leq 1 \text{ implies that } \Pr \left( \bigcap_{i=1}^{m} \bar{\mathcal{E}}_i \right) \geq (1 - 2p)^m > 0.
\]
Proof of Theorem 14: We randomly color the elements of $A$ Red and Blue. Let $\mathcal{E}_i$ be the event that $A_i$ is mono-colored. Clearly, $\Pr(\mathcal{E}_i) \leq 2^{-(k-1)}$. Thus,

$$p \leq 2^{-(k-1)}.$$ 

The degree of vertex $i$ of $\Gamma$ is the number of $j$ such that $A_i \cap A_j \neq \emptyset$. So, by assumption,

$$d \leq 2^{k-3}.$$ 

Theorem 15 implies that $\Pr (\bigcap_{i=1}^n \bar{\mathcal{E}}_i) > 0$ and so the required coloring exists.
Theorem

Let $G = (V, E)$ be an $r$-regular graph. If $r$ is sufficiently large, then $E$ can be partitioned into $E_1, E_2$ so that if $G_i = (V, E_i), i = 1, 2$ then

$$\frac{r}{2} - (20r \log r)^{1/2} \leq \delta(G_i) \leq \Delta(G_i) \leq \frac{r}{2} + (20r \log r)^{1/2}.$$ 

Proof: We randomly partition the edges of $G$ by independently placing $e$ into $E_1 E_1$ with probability $1/2$. For $v \in V$, we let $\mathcal{E}_v$ be the event that the degree $d_1(v)$ in $G_1$ satisfies

$$d_1(v) \notin \left[\frac{r}{2} - (3r \log r)^{1/2}, \frac{r}{2} + (3r \log r)^{1/2}\right].$$
It follows from Hoeffding’s Inequality - I with $t = (3r \log r)^{1/2}$ that

$$
\Pr(E_v) \leq 2e^{-t^2/2r} = 2r^{-3/2}. \quad (7)
$$

Furthermore, $E_v$ is independent of the events $E_w$ for vertices $w$ at distance 2 or more from $v$ in $G$. Thus,

$$
d \leq r.
$$

Clearly, $4 \cdot 2r^{-3/2} \cdot r \leq 1$ for $r$ large and the result follows from Theorem 15. I.e. $\Pr(\bigcap_{v \in V} \bar{E}_v) > 0$ which implies that there exists a partition where none of the events $E_v, v \in V$ occur.
For the next application, let $D = (V, E)$ be a $k$-regular digraph. By this we mean that each vertex has exactly $k$ in-neighbors and $k$ out-neighbors.

**Theorem**

Every $k$-regular digraph has a collection of $\left\lfloor \frac{k}{4 \log k} \right\rfloor$ vertex-disjoint cycles.

**Proof:** Let $r = \left\lfloor \frac{k}{4 \log k} \right\rfloor$ and color the vertices of $D$ with colors $[r]$. For $v \in V$, let $E_v$ be the event that there is a color missing at the out-neighbors of $v$. We will show that $\Pr(\bigcap_{v \in V} \overline{E_v}) > 0$.

Suppose then that none of the events $E_v$, $v \in V$ occur. Consider the graph $D_j$ induced by a single color $j \in [r]$. Note that $D_j$ is not the empty graph. Let $P_j = (v_1, v_2, \ldots, v_m)$ be a longest directed path in $D_j$. Let $w$ be an out-neighbor of $v_m$ of color $j$. We must have $w \in \{v_1, \ldots, v_m\}$, else $P_j$ is not a longest path in $D_j$. Thus each $D_j, j \in [r]$ contains a cycle and these cycles are vertex disjoint.
We first estimate

\[ \Pr(\mathcal{E}_v) \leq k \left(1 - \frac{1}{r}\right)^k \leq ke^{-k/r} \leq ke^{-4 \log k} = k^{-3}. \]

On the other hand, if \( N^+(v) \) denotes the out-neighbors of \( v \) plus \( v \) then \( \mathcal{E}_v \) is independent of all events \( \mathcal{E}_w \) for which \( N^+(v) \cap N^+(w) = \emptyset \). It follows that

\[ d \leq k^2. \]

To apply Theorem 15 we need to have \( 4k^{-3}k^2 \leq 1 \). This is true for \( k \geq 4 \). For \( k \leq 3 \) we have \( r = 1 \) and the local lemma is not needed.
Let $\mathcal{P}_n = \{A : A \subseteq [n]\}$ denote the *power set* of $[n]$.

$A \subseteq \mathcal{P}_n$ is a *Sperner* family if $A, B \in A$ implies that $A \not\subseteq B$ and $B \not\subseteq A$.

**Theorem**

*If* $A \subseteq \mathcal{P}_n$ *is a Sperner family* $|A| \leq \binom{n}{\lfloor n/2 \rfloor}$.

**Proof**

We will show that

$$\sum_{A \in A} \frac{1}{\binom{n}{|A|}} \leq 1.$$ (8)

Now $\binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor}$ for all $k$ and so

$$1 \geq \sum_{A \in A} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{|A|}{\binom{n}{\lfloor n/2 \rfloor}}.$$
Proof of (8): Let $\pi$ be a random permutation of $[n]$.

For a set $A \in \mathcal{A}$ let $\mathcal{E}_A$ be the event

$$\{\pi(1), \pi(2), \ldots, \pi(|A|)\} = A.$$ 

If $A, B \in \mathcal{A}$ then the events $\mathcal{E}_A, \mathcal{E}_B$ are disjoint.

So

$$\sum_{A \in \mathcal{A}} \Pr(\mathcal{E}_A) \leq 1.$$ 

On the other hand, if $A \in \mathcal{A}$ then

$$\Pr(\mathcal{E}_A) = \frac{|A|!(n-|A|)!}{n!} = \frac{1}{\binom{n}{|A|}}$$

and (18) follows.
The set of all sets of size \( \lfloor n/2 \rfloor \) is a Sperner family and so the bound in the above theorem is best possible.

Inequality (8) can be generalised as follows: Let \( s \geq 1 \) be fixed. Let \( \mathcal{A} \) be a family of subsets of \([n]\) such that there do not exist distinct \( A_1, A_2, \ldots, A_{s+1} \in \mathcal{A} \) such that \( A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{s+1} \).

**Theorem**

\[
\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq s.
\]

**Proof**  
Let \( \pi \) be a random permutation of \([n]\).

Let \( \mathcal{E}(A) \) be the event \( \{\pi(1), \pi(2), \ldots, \pi(|A|) = A\} \).
Let
\[ Z_i = \begin{cases} 
1 & \text{ } \mathcal{E}(A_i) \text{ occurs.} \\
0 & \text{otherwise.} 
\end{cases} \]

and let \( Z = \sum_i Z_i \) be the number of events \( \mathcal{E}(A_i) \) that occur.

Now our family is such that \( Z \leq s \) for all \( \pi \) and so
\[
E(Z) = \sum_i E(Z_i) = \sum_i \Pr(\mathcal{E}(A_i)) \leq s.
\]

On the other hand, \( A \in \mathcal{A} \) implies that \( \Pr(\mathcal{E}(A)) = \frac{1}{\binom{n}{|A|}} \) and the required inequality follows. \( \square \)
9/29/2021
Extremal Problems

**Intersecting Families** A family $\mathcal{A} \subseteq \mathcal{P}_n$ is an *intersecting* family if $A, B \in \mathcal{A}$ implies $A \cap B \neq \emptyset$.

**Theorem** If $\mathcal{A}$ is an intersecting family then $|\mathcal{A}| \leq 2^{n-1}$.

**Proof** Pair up each $A \in \mathcal{P}_n$ with its complement $A^c = [n] \setminus A$. This gives us $2^{n-1}$ pairs altogether. Since $\mathcal{A}$ is intersecting it can contain at most one member of each pair.

If $\mathcal{A} = \{A \subseteq [n] : 1 \in A\}$ then $\mathcal{A}$ is intersecting and $|\mathcal{A}| = 2^{n-1}$ and so the above theorem is best possible.
Theorem

If $\mathcal{A}$ is an intersecting family and $A \in \mathcal{A}$ implies that $|A| = k \leq \lfloor n/2 \rfloor$ then

$$|\mathcal{A}| \leq \binom{n-1}{k-1}$$

Proof

If $\pi$ is a permutation of $[n]$ and $A \subseteq [n]$ let

$$\theta(\pi, A) = \begin{cases} 
1 & \exists s : \{\pi(s), \pi(s+1), \ldots, \pi(s+k-1)\} = A \\
0 & \text{otherwise}
\end{cases}$$

where $\pi(i) = \pi(i-n)$ if $i > n$.

We will show that for any permutation $\pi$,

$$\sum_{A \in \mathcal{A}} \theta(\pi, A) \leq k. \quad (9)$$
Assume (9). We first observe that if $\pi$ is a random permutation then

$$E(\theta(\pi, A)) = n \frac{k!(n-k)!}{n!} = \frac{k}{\binom{n-1}{k-1}}$$

and so, from (9),

$$k \geq E\left( \sum_{A \in \mathcal{A}} \theta(\pi, A) \right) = \sum_{A \in \mathcal{A}} \frac{k}{\binom{|A|-1}{n-1}}$$

Hence

$$|\mathcal{A}| \leq \binom{n-1}{k-1}$$
Assume w.l.o.g. that $\pi$ is the identity permutation.

Let $A_t = \{t, t + 1, \ldots, t + k - 1\}$ and suppose that $A_s \in \mathcal{A}$.

All of the other sets $A_t$ that intersect $A_s$ can be partitioned into pairs $A_{s-i}, A_{s+k-i}$, $1 \leq i \leq k - 1$ and the members of each pair are disjoint. Thus $\mathcal{A}$ can contain at most one from each pair. This verifies (9).
Kraft’s Inequality

Let $x_1, x_2, \ldots, x_m$ be a collection of sequences over an alphabet $\Sigma$ of size $s$. Let $x_i$ have length $n_i$ and let $n = \max\{n_1, n_2, \ldots, n_m\}$.

Assume next that no sequence is a prefix of any other sequence: Sequence $x_i = a_1 a_2 \cdots a_{n_i}$ is a prefix of $x_j = b_1 b_2 \cdots b_{n_j}$ if $a_i = b_i$ for $i = 1, 2, \ldots, n_i$.

**Theorem**

$$\sum_{i=1}^{m} r^{-n_i} \leq 1.$$
Proof: Let $x$ be a random sequence of length $n$. Let $\mathcal{E}_i$ be the event $x_i$ is a prefix of $x$. Then

(a) $\Pr(\mathcal{E}_i) = r^{-n_i}$.

(b) The event $\mathcal{E}_i$, $i = 1, 2, \ldots, m$ are disjoint. (If $\mathcal{E}_i$ and $\mathcal{E}_j$ both occur and $n_i \leq n_j$ then $x_i$ is a prefix of $x_j$.

Property (b) implies that

$$\Pr \left( \bigcup_{i=1}^{m} \mathcal{E}_i \right) = \Pr(\mathcal{E}_1) + \Pr(\mathcal{E}_2) + \cdots + \Pr(\mathcal{E}_m) \leq 1.$$ 

The theorem now follows from Property (a). $\square$
A sunflower of size \( r \) is a family of sets \( A_1, A_2, \ldots, A_r \) such that every element that belongs to more than one of the sets belongs to all of them.

Let \( f(k, r) \) be the maximum size of a family of \( k \)-sets without a sunflower of size \( r \).

**Theorem** \( f(k, r) \leq (r - 1)^k k! \).

**Proof** Let \( \mathcal{F} \) be a family of \( k \)-sets without a sunflower of size \( r \). Let \( A_1, A_2, \ldots, A_t \) be a maximum subfamily of pairwise disjoint subsets in \( \mathcal{F} \).

Since a family of pairwise disjoint is a sunflower, we must have \( t < r \).
Now let $A = \bigcup_{i=1}^{t} A_i$. For every $a \in A$ consider the family $\mathcal{F}_a = \{ S \setminus \{a\} : S \in \mathcal{F}, a \in S \}$.

Now the size of $A$ is at most $(r - 1)k$.

The size of each $\mathcal{F}_a$ is at most $f(k - 1, r)$. This is because a sunflower in $\mathcal{F}_a$ is a sunflower in $\mathcal{F}$.

So,

$$f(k, r) \leq (r - 1)k \times f(k - 1, r) \leq (r - 1)k \times (r - 1)^{k-1}(k - 1)!,$$

by induction. \qed
10/1/2021
**Odd Town** In order to cut down the number of committees a town of $n$ people has instituted the following rules:

(a) Each club shall have an odd number of members.

(b) Each pair of clubs shall share an even number of members.

**Theorem**

*With these rules, there are at most $n$ clubs.*
Proof Suppose that the clubs are $C_1, C_2, \ldots, C_m \subseteq [n]$.

Let $\vec{v}_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,n})$ denote the incidence vector of $C_i$ for $1 \leq i \leq m$ i.e. $v_{i,j} = 1$ iff $j \in C_i$. We treat these vectors as being over the two element field $\mathbb{F}_2$.

We claim that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are linearly independent and the theorem will follow.

The rules imply that (i) $\vec{v}_i \cdot \vec{v}_i = 1$ and (ii) $\vec{v}_i \cdot \vec{v}_j = 0$ for $1 \leq i \neq j \leq m$.
(Remember that we are working over $\mathbb{F}_2$.)
Suppose then that

\[ c_1 \bar{\nu}_1 + c_2 \bar{\nu}_2 + \cdots + c_m \bar{\nu}_m = 0. \]

We show that \( c_1 = c_2 = \cdots = c_m = 0. \)

Indeed, we have

\[
0 = \bar{\nu}_j \cdot (c_1 \bar{\nu}_1 + c_2 \bar{\nu}_2 + \cdots + c_m \bar{\nu}_m) \\
= c_1 \bar{\nu}_1 \cdot \bar{\nu}_j + c_2 \bar{\nu}_2 \cdot \bar{\nu}_j + \cdots + c_m \bar{\nu}_m \cdot \bar{\nu}_j \\
= c_j,
\]

for \( j = 1, 2, \ldots, m. \) □
Decomposing \( K_n \) into bipartite subgraphs: here we show

**Theorem**

If \( G_k, k = 1, 2, \ldots, m \) is a collection of complete bipartite graphs with vertex partitions \( A_k, B_k \), such that every edge of \( K_n \) is in exactly one subgraph, then \( m \geq n - 1 \). (Note that \( A_k \cap B_k = \emptyset \) here.)

**Proof**  
This is tight since we can take \( A_k = \{k\}, B_k = \{k + 1, \ldots, n\} \) for \( k = 1, 2, \ldots, n - 1 \).

Define \( n \times n \) matrices \( M_k \) where \( M_k(i, j) = 1 \) if \( i \in A_k, j \in B_k \) and \( M_k(i, j) = 0 \) otherwise.

Let \( S = M_1 + M_2 + \cdots + M_m \). Then \( S + S^T = J_n - I_n \) where \( I_n \) is the identity matrix and \( J_n \) is the all ones matrix.
We show next that \( \text{rank}(S) \geq n - 1 \) and then the theorem follows from

\[
\text{rank}(S) \leq \text{rank}(M_1) + \text{rank}(M_2) + \cdots + \text{rank}(M_m) \leq m.
\]

Suppose then that \( \text{rank}(S) \leq n - 2 \) so that there exists a non-zero solution \( x = (x_1, x_2, \ldots, x_n)^T \) to the system of equations

\[
Sx = 0, \quad \sum_{i=1}^{n} x_i = 0.
\]

But then, \( J_n x = 0 \) and \( S^T x = -x \) and \( -|x|^2 = -x^T S^T x = 0 \), contradiction. □
Nonuniform Fisher Inequality;

Theorem

Let $C_1, C_2, \ldots, C_m$ be distinct subsets of $[n]$ such that for every $i \neq j$ we have $|C_i \cap C_j| = s$ where $1 \leq s < n$. Then $m \leq n$.

Proof

If $|C_1| = s$ then $C_i \supset C_1, i = 2, 3, \ldots, m$ and the sets $C_i \setminus C_1$ are pairwise disjoint for $i \geq 2$.

It follows in this case that $m \leq 1 + n - s \leq n$.

Assume from now on that $c_i = |C_i| - s > 0$ for $i \in [m]$. 
Let $M$ be the $m \times n$ 0/1 matrix where $M(i, j) = 1$ iff $j \in C_i$.

Let

$$A = MM^T = sJ + D$$

where $J$ is the $m \times m$ all 1’s matrix and $D$ is the diagonal matrix, where $D(i, i) = c_i$.

We show that $A$ and hence $M$ has rank $m$, implying that $m \leq n$ as claimed.

We will in fact show that $x^TAx > 0$ for all $0 \neq x \in \mathbb{R}^m$. This means that $Ax \neq 0$ when $x \neq 0$. 
If $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T$ then

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = s(x_1 + x_2 + \cdots + x_m)^2 + \sum_{i=1}^{m} c_i x_i^2 > 0.$$
10/4/2021
We have two disks, each partitioned into 200 sectors of the same size. 100 of the sectors of Disk 1 are coloured Red and 100 are colored Blue. The 200 sectors of Disk 2 are arbitrarily coloured Red and Blue.

It is always possible to place Disk 2 on top of Disk 1 so that the centres coincide, the sectors line up and at least 100 sectors of Disk 2 have the same colour as the sector underneath them.

Fix the position of Disk 1. There are 200 positions for Disk 2 and let \( q_i \) denote the number of matches if Disk 2 is placed in position \( i \). Now for each sector of Disk 2 there are 100 positions \( i \) in which the colour of the sector underneath it coincides with its own.
Therefore

\[ q_1 + q_2 + \cdots + q_{200} = 200 \times 100 \]  \hspace{1cm} (10)

and so there is an \( i \) such that \( q_i \geq 100 \).

Explanation of (19).
Consider 0-1 \( 200 \times 200 \) matrix \( A(i, j) \) where \( A(i, j) = 1 \) iff sector \( j \) lies on top of a sector with the same colour when in position \( i \). Row \( i \) of \( A \) has \( q_i \) 1’s and column \( j \) of \( A \) has 100 1’s. The LHS of (19) counts the number of 1’s by adding rows and the RHS counts the number of 1’s by adding columns.
Alternative solution: Place Disk 2 randomly on Disk 1 so that the sectors align. For \( i = 1, 2, \ldots, 200 \) let

\[
X_i = \begin{cases} 
1 & \text{sector } i \text{ of disk 2 is on sector of disk 1 of same color} \\
0 & \text{otherwise}
\end{cases}
\]

We have

\[
E(X_i) = \frac{1}{2} \quad \text{for } i = 1, 2, \ldots, 200.
\]

So if \( X = X_1 + \cdots + X_{200} \) is the number of sectors sitting above sectors of the same color, then \( E(X) = 100 \) and there must exist at least one way to achieve 100.
Theorem

*(Erdős-Szekeres)* An arbitrary sequence of integers \((a_1, a_2, \ldots, a_{k^2+1})\) contains a monotone subsequence of length \(k + 1\).

**Proof.** Let \((a_i, a^1_i, a^2_i, \ldots, a^{\ell-1}_i)\) be the longest *monotone increasing* subsequence of \((a_1, \ldots, a_{k^2+1})\) that starts with \(a_i, (1 \leq i \leq k^2 + 1)\), and let \(\ell(a_i)\) be its length.

If for some \(1 \leq i \leq k^2 + 1\), \(\ell(a_i) \geq k + 1\), then \((a_i, a^1_i, a^2_i, \ldots, a^{\ell-1}_i)\) is a monotone increasing subsequence of length \(\geq k + 1\).

So assume that \(\ell(a_i) \leq k\) holds for every \(1 \leq i \leq k^2 + 1\).
Consider $k$ holes $1, 2, \ldots, k$ and place $i$ into hole $\ell(a_i)$.

There are $k^2 + 1$ subsequences and $\leq k$ non-empty holes (different lengths), so by the pigeon-hole principle there will exist $\ell^*$ such that there are (at least) $k + 1$ indices $i_1 < i_2 < \cdots < i_{k+1}$ such that $\ell(a_{i_t}) = \ell^*$ for $1 \leq t \leq k + 1$.

Then we must have $a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_{k+1}}$.

Indeed, assume to the contrary that $a_{i_m} < a_{i_n}$ for some $1 \leq m < n \leq k + 1$. Then $a_{i_m} \leq a_{i_n} \leq a_{i_n}^1 \leq a_{i_n}^2 \leq \cdots \leq a_{i_n}^{\ell^* - 1}$, i.e., $\ell(a_{i_m}) \geq \ell^* + 1$, a contradiction. □
The sequence

\[ n, n-1, \ldots, 1, 2n, 2n-1, \ldots, n+1, \ldots, n^2, n^2-1, \ldots, n^2-n+1 \]

has no monotone subsequence of length \( n+1 \) and so the Erdős-Szekeres result is best possible.
Pigeon Hole Principle

Let $P_1, P_2, \ldots, P_n$ be $n$ points in the unit square $[0, 1]^2$. We will show that there exist $i, j, k \in [n]$ such that the triangle $P_iP_jP_k$ has area

$$\leq \frac{1}{2\left(\sqrt{(n - 1)/2}\right)^2} \sim \frac{1}{n}$$

for large $n$. 
Let \( m = \left\lfloor \sqrt{(n - 1)/2} \right\rfloor \) and divide the square up into \( m^2 < \frac{n}{2} \) subsquares. By the pigeonhole principle, there must be a square containing \( \geq 3 \) points. Let 3 of these points be \( P_i P_j P_k \). The area of the corresponding triangle is at most one half of the area of an individual square.
Suppose we 2-colour the edges of $K_6$ of Red and Blue. There must be either a Red triangle or a Blue triangle.

This is not true for $K_5$. 
There are 3 edges of the same colour incident with vertex 1, say (1,2), (1,3), (1,4) are Red. Either (2,3,4) is a blue triangle or one of the edges of (2,3,4) is Red, say (2,3). But the latter
Ramsey’s Theorem

For all positive integers $k, \ell$ there exists $R(k, \ell)$ such that if $N \geq R(k, \ell)$ and the edges of $K_N$ are coloured Red or Blue then either there is a “Red $k$-clique” or there is a “Blue $\ell$-clique. A clique is a complete subgraph and it is Red if all of its edges are coloured red etc.

$$R(1, k) = R(k, 1) = 1$$
$$R(2, k) = R(k, 2) = k$$
**Theorem**

\[ R(k, \ell) \leq R(k, \ell - 1) + R(k - 1, \ell). \]

**Proof** Let \( N = R(k, \ell - 1) + R(k - 1, \ell) \).

\[ V_R = \{ x : (1, x) \text{ is coloured Red} \} \text{ and } V_B = \{ x : (1, x) \text{ is} \]
\[ |V_R| \geq R(k - 1, \ell) \text{ or } |V_B| \geq R(k, \ell - 1). \]

Since

\[ |V_R| + |V_B| = N - 1 = R(k, \ell - 1) + R(k - 1, \ell) - 1. \]

Suppose for example that \( |V_R| \geq R(k - 1, \ell) \). Then either \( V_R \) contains a Blue \( \ell \)-clique – done, or it contains a Red \( k - 1 \)-clique \( K \). But then \( K \cup \{1\} \) is a Red \( k \)-clique. Similarly, if \( |V_B| \geq R(k, \ell - 1) \) then either \( V_B \) contains a Red \( k \)-clique – done, or it contains a Blue \( \ell - 1 \)-clique \( L \) and then \( L \cup \{1\} \) is a Blue \( \ell \)-clique. \( \square \)
Ramsey Theory

**Theorem**

\[ R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}. \]

**Proof**

Induction on \( k + \ell \). True for \( k + \ell \leq 5 \) say. Then

\[
R(k, \ell) \leq R(k, \ell - 1) + R(k - 1, \ell) \\
\leq \binom{k + \ell - 3}{k - 1} + \binom{k + \ell - 3}{k - 2} \\
= \binom{k + \ell - 2}{k - 1}.
\]

\[ \square \]

So, for example,

\[ R(k, k) \leq \binom{2k - 2}{k - 1}. \]
Theorem

\[ R(k, k) > 2^{k/2} \]

**Proof**  We must prove that if \( n \leq 2^{k/2} \) then there exists a Red-Blue colouring of the edges of \( K_n \) which contains no Red \( k \)-clique and no Blue \( k \)-clique. We can assume \( k \geq 4 \) since we know \( R(3, 3) = 6 \).

We show that this is true with positive probability in a *random* Red-Blue colouring. So let \( \Omega \) be the set of all Red-Blue edge colourings of \( K_n \) with uniform distribution. Equivalently we independently colour each edge Red with probability 1/2 and Blue with probability 1/2.
Let $\mathcal{E}_R$ be the event: \{There is a Red $k$-clique\} and $\mathcal{E}_B$ be the event: \{There is a Blue $k$-clique\}.

We show

$$\Pr(\mathcal{E}_R \cup \mathcal{E}_B) < 1.$$ 

Let $C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$ be the vertices of the $N$ $k$-cliques of $K_n$.

Let $\mathcal{E}_{R,j}$ be the event: \{$C_j$ is Red\} and let $\mathcal{E}_{B,j}$ be the event: \{$C_j$ is Blue\}.
\[ \Pr(\mathcal{E}_R \cup \mathcal{E}_B) \leq \Pr(\mathcal{E}_R) + \Pr(\mathcal{E}_B) = 2\Pr(\mathcal{E}_R) \]

\[ = 2\Pr \left( \bigcup_{j=1}^{N} \mathcal{E}_{R,j} \right) \leq 2 \sum_{j=1}^{N} \Pr(\mathcal{E}_{R,j}) \]

\[ = 2 \sum_{j=1}^{N} \left( \frac{1}{2} \right)^{C_{k}} = 2 \left( \frac{n}{k} \right) \left( \frac{1}{2} \right)^{\binom{k}{2}} \]

\[ \leq 2 \frac{n^k}{k!} \left( \frac{1}{2} \right)^{\binom{k}{2}} \]

\[ \leq 2 \frac{2^{k^2/2}}{k!} \left( \frac{1}{2} \right)^{\binom{k}{2}} \]

\[ \leq \frac{2^{1+k/2}}{k!} \]

\[ \leq \frac{1}{k!} \]

\[ < 1. \]
Very few of the Ramsey numbers are known exactly. Here are a few known values.

\[
\begin{align*}
R(3,3) &= 6 \\
R(3,4) &= 9 \\
R(4,4) &= 18 \\
R(4,5) &= 25 \\
43 &\leq R(5,5) \leq 49
\end{align*}
\]
Schur’s Theorem

Let $r_k = N(3, 3, \ldots, 3; 2)$ be the smallest $n$ such that if we $k$-color the edges of $K_n$ then there is a mono-chromatic triangle.

**Theorem**

For all partitions $S_1, S_2, \ldots, S_k$ of $[r_k]$, there exist $i$ and $x, y, z \in S_i$ such that $x + y = z$.

**Proof**

Given a partition $S_1, S_2, \ldots, S_k$ of $[n]$ where $n \geq r_k$ we define a coloring of the edges of $K_n$ by coloring $(u, v)$ with color $j$ where $|u - v| \in S_j$.

There will be a mono-chromatic triangle i.e. there exist $j$ and $x < y < z$ such that $u = y - x$, $v = z - x$, $w = z - y \in S_j$. But $u + v = w$. 

 Covered so far
A set of points $X$ in the plane is in general position if no 3 points of $X$ are collinear.

**Theorem**

If $n \geq N(k, k; 3)$ and $X$ is a set of $n$ points in the plane which are in general position then $X$ contains a $k$-subset $Y$ which form the vertices of a convex polygon.

**Proof**  
We first observe that if every 4-subset of $Y \subseteq X$ forms a convex quadrilateral then $Y$ itself induces a convex polygon.

Now label the points in $S$ from $X_1$ to $X_n$ and then color each triangle $T = \{X_i, X_j, X_k\}$, $i < j < k$ as follows: If traversing triangle $X_iX_jX_k$ in this order goes round it clockwise, color $T$ Red, otherwise color $T$ Blue.
Now there must exist a $k$-set $T$ such that all triangles formed from $T$ have the same color. All we have to show is that $T$ does not contain the following configuration:
Assume w.l.o.g. that $a < b < c$ which implies that $X_iX_jX_k$ is colored Blue.

All triangles in the previous picture are colored Blue.

So the possibilities are

$$
\begin{array}{c}
\text{adc} \\
\text{bcd} \\
\text{dbc}
\end{array}
\Rightarrow
\begin{array}{c}
\text{abd} \\
\text{dab}
\end{array}
$$

and all are impossible.
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We define $r(H_1, H_2)$ to be the minimum $n$ such that in a Red-Blue coloring of the edges of $K_n$ there is either (i) a Red copy of $H_1$ or (ii) a Blue copy of $H_2$.

As an example, consider $r(P_3, P_3)$ where $P_t$ denotes a path with $t$ edges.

We show that

$$r(P_3, P_3) = 5.$$ 

$R(P_3, P_3) > 4$: We color edges incident with 1 Red and the remaining edges $\{(2, 3), (3, 4), (4, 1)\}$ Blue. There is no mono-chromatic $P_3$. 
\textbf{Ramsey Theory}

$R(P_3, P_3) \leq 5$: There must be two edges of the same color incident with 1.

Assume then that $(1, 2), (1, 3)$ are both Red.

If any of $(2, 4), (2, 5), (3, 4), (3, 5)$ are Red then we have a Red $P_3$.

If all four of these edges are Blue then $(4, 2, 5, 3)$ is Blue.
We show next that $r(K_{1,s}, P_t) \leq s + t$. Here $K_{1,s}$ is a star: i.e. a vertex $v$ and $t$ incident edges.

Let $n = s + t$. If there is no vertex of Red degree $s$ then the minimum degree in the graph induced by the Blue edges is at least $t$.

We then note that a graph of minimum degree $\delta$ contains a path of length $\delta$. 

Covered so far
A partially ordered set or **poset** is a set $P$ and a binary relation $\leq$ such that for all $a, b, c \in P$

1. $a \leq a$ (reflexivity).
2. $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).
3. $a \leq b$ and $b \leq a$ implies $a = b$. (anti-symmetry).

**Examples**

1. $P = \{1, 2, \ldots, \}$ and $a \leq b$ has the usual meaning.
2. $P = \{1, 2, \ldots, \}$ and $a \leq b$ if $a$ divides $b$.
3. $P = \{A_1, A_2, \ldots, A_m\}$ where the $A_i$ are sets and $\leq = \subseteq$. 
A pair of elements $a, b$ are **comparable** if $a \leq b$ or $b \leq a$. Otherwise they are **incomparable**.

A poset without incomparable elements (Example 1) is a linear or total order.

We write $a < b$ if $a \leq b$ and $a \neq b$.

A **chain** is a sequence $a_1 < a_2 < \cdots < a_s$.

A set $A$ is an **anti-chain** if every pair of elements in $A$ are incomparable.

Thus a Sperner family is an anti-chain in our third example.
Theorem

Let $P$ be a finite poset, then
\[
\min \{ m : \exists \text{ anti-chains } A_1, A_2, \ldots, A_\mu \text{ with } P = \bigcup_{i=1}^\mu A_i \} = \max \{|C| : A \text{ is a chain} \}.
\]

The minimum number of anti-chains needed to cover $P$ is at least the size of any chain, since a chain can contain at most one element from each anti-chain.
We prove the converse by induction on the maximum length $\mu$ of a chain. We have to show that $P$ can be partitioned into $\mu$ anti-chains.

If $\mu = 1$ then $P$ itself is an anti-chain and this provides the basis of the induction.

So now suppose that $C = x_1 < x_2 < \cdots < x_\mu$ is a maximum length chain and let $A$ be the set of maximal elements of $P$.

(An element is $x$ maximal if $\not\exists y$ such that $y > x$.)

$A$ is an anti-chain.
Now consider $P' = P \setminus A$. $P'$ contains no chain of length $\mu$. If it contained $y_1 < y_2 < \cdots < y_\mu$ then since $y_\mu \notin A$, there exists $a \in A$ such that $P$ contains the chain $y_1 < y_2 < \cdots < y_\mu < a$, contradiction.

Thus the maximum length of a chain in $P'$ is $\mu - 1$ and so it can be partitioned into anti-chains $A_1 \cup A_2 \cup \cdots A_{\mu-1}$. Putting $A_\mu = A$ completes the proof. \qed
Suppose that $C_1, C_2, \ldots, C_m$ are a collection of chains such that $P = \bigcup_{i=1}^{m} C_i$.

Suppose that $A$ is an anti-chain. Then $m \geq |A|$ because if $m < |A|$ then by the pigeon-hole principle there will be two elements of $A$ in some chain.

**Theorem**

(Dilworth) Let $P$ be a finite poset, then

$$\min\{m : \exists \text{ chains } C_1, C_2, \ldots, C_m \text{ with } P = \bigcup_{i=1}^{m} C_i\} = \max\{|A| : A \text{ is an anti-chain}\}.$$
Intervals Problem

$l_1, l_2, \ldots, l_{mn+1}$ are closed intervals on the real line i.e. $l_j = [a_j, b_j]$ where $a_j \leq b_j$ for $1 \leq j \leq mn + 1$.

**Theorem**

Either (i) there are $m + 1$ intervals that are pair-wise disjoint or (ii) there are $n + 1$ intervals with a non-empty intersection.

Define a partial ordering $\preceq$ on the intervals by $l_r \preceq l_s$ iff $b_r \leq a_s$. Suppose that $l_{i_1}, l_{i_2}, \ldots, l_{i_t}$ is a collection of pair-wise disjoint intervals. Assume that $a_{i_1} < a_{i_2} \cdots < a_{i_t}$. Then $l_{i_1} < l_{i_2} \cdots < l_{i_t}$ form a chain and conversely a chain of length $t$ comes from a set of $t$ pair-wise disjoint intervals.

So if (i) does not hold, then the maximum length of a chain is $m$. 

Covered so far
This means that the minimum number of chains needed to cover the poset is at least \( \lceil \frac{mn+1}{m} \rceil = n + 1 \).

Dilworth’s theorem implies that there must exist an anti-chain \( \{ I_{j_1}, I_{j_2}, \ldots, I_{j_{n+1}} \} \).

Let \( a = \max \{ a_{j_1}, a_{j_2}, \ldots, a_{j_{n+1}} \} \) and \( b = \min \{ b_{j_1}, b_{j_2}, \ldots, b_{j_{n+1}} \} \).

We must have \( a \leq b \) else the two intervals giving \( a, b \) are disjoint.

But then every interval of the anti-chain contains \([a, b]\).
Suppose that $C_1, C_2, \ldots, C_m$ are a collection of chains such that $P = \bigcup_{i=1}^{m} C_i$.

Suppose that $A$ is an anti-chain. Then $m \geq |A|$ because if $m < |A|$ then by the pigeon-hole principle there will be two elements of $A$ in some chain.

**Theorem**

*(Dilworth)* Let $P$ be a finite poset, then

$$\min \{ m : \exists \text{ chains } C_1, C_2, \ldots, C_m \text{ with } P = \bigcup_{i=1}^{m} C_i \} = \max \{ |A| : A \text{ is an anti-chain} \}.$$
We have already argued that \( \max\{|A|\} \leq \min\{m\} \).

We will prove there is equality here by induction on \( |P| \).

The result is trivial if \( |P| = 0 \).

Now assume that \( |P| > 0 \) and that \( \mu \) is the maximum size of an anti-chain in \( P \). We show that \( P \) can be partitioned into \( \mu \) chains.

Let \( C = x_1 < x_2 < \cdots < x_p \) be a maximal chain in \( P \) i.e. we cannot add elements to it and keep it a chain.

**Case 1** Every anti-chain in \( P \setminus C \) has \( \leq \mu - 1 \) elements. Then by induction \( P \setminus C = \bigcup_{i=1}^{\mu-1} C_i \) and then \( P = C \cup \bigcup_{i=1}^{\mu-1} C_i \) and we are done.
Case 2

There exists an anti-chain $A = \{a_1, a_2, \ldots, a_\mu\}$ in $P \setminus C$. Let

- $P^- = \{x \in P : x \preceq a_i \text{ for some } i\}$.
- $P^+ = \{x \in P : x \succeq a_i \text{ for some } i\}$.

Note that

1. $P = P^- \cup P^+$. Otherwise there is an element $x$ of $P$ which is incomparable with every element of $A$ and so $\mu$ is not the maximum size of an anti-chain.

2. $P^- \cap P^+ = A$. Otherwise there exists $x, i, j$ such that $a_i < x < a_j$ and so $A$ is not an anti-chain.

3. $x_p \notin P^-$. Otherwise $x_p < a_i$ for some $i$ and the chain $C$ is not maximal.
Applying the inductive hypothesis to $P^-$ ($|P^-| < |P|$ follows from 3) we see that $P^-$ can be partitioned into $\mu$ chains $C_1^-, C_2^-, \ldots, C_{\mu}^-$. 

Now the elements of $A$ must be distributed one to a chain and so we can assume that $a_i \in C_i^-$ for $i = 1, 2, \ldots, \mu$.

$a_i$ must be the maximum element of chain $C_i^-$, else the maximum of $C_i^-$ is in $(P^- \cap P^+) \setminus A$, which contradicts 2.

Applying the same argument to $P^+$ we get chains $C_1^+, C_2^+, \ldots, C_{\mu}^+$ with $a_i$ as the minimum element of $C_i^+$ for $i = 1, 2, \ldots, \mu$.

Then from 2 we see that $P = C_1 \cup C_2 \cup \cdots \cup C_\mu$ where $C_i = C_i^- \cup C_i^+$ is a chain for $i = 1, 2, \ldots, \mu$. \[\square\]
(i) Another proof of

**Theorem**

*Erdős and Szekeres*

$a_1, a_2, \ldots, a_{n^2+1}$ contains a monotone subsequence of length $n+1$.

Let $P = \{(i, a_i) : 1 \leq i \leq n^2 + 1\}$ and let say $(i, a_i) \preceq (j, a_j)$ if $i < j$ and $a_i \leq a_j$.

A chain in $P$ corresponds to a monotone increasing subsequence. So, suppose that there are no monotone increasing sequences of length $n + 1$. Then any cover of $P$ by chains requires at least $n + 1$ chains and so, by Dilworths theorem, there exists an anti-chain $A$ of size $n + 1$. 
Let $A = \{(i_t, a_{i_t}) : 1 \leq t \leq n + 1\}$ where $i_1 < i_2 \leq \cdots < i_{n+1}$.

Observe that $a_{i_t} > a_{i_{t+1}}$ for $1 \leq t \leq n$, for otherwise $(i_t, a_{i_t}) \preceq (i_{t+1}, a_{i_{t+1}})$ and $A$ is not an anti-chain.

Thus $A$ defines a monotone decreasing sequence of length $n + 1$. \qed
Matchings in bipartite graphs

Re-call that a matching is a set of vertex disjoint edges.
Let $G = (A \cup B, E)$ be a bipartite graph with bipartition $A, B$. For $S \subseteq A$ let $N(S) = \{ b \in B : \exists a \in S, (a, b) \in E \}$.

Clearly, $|M| \leq |A|, |B|$ for any matching $M$ of $G$. 

Covered so far
Theorem

(Hall) \( G \) contains a matching of size \( |A| \) iff

\[ |N(S)| \geq |S| \quad \forall S \subseteq A. \]

\( N(\{a_1, a_2, a_3\}) = \{b_1, b_2\} \) and so at most 2 of \( a_1, a_2, a_3 \) can be saturated by a matching.
If $G$ contains a matching $M$ of size $|A|$ then
$M = \{(a, f(a)) : a \in A\}$, where $f : A \rightarrow B$ is a 1-1 function.

But then,

$$|N(S)| \geq |f(S)| = S$$

for all $S \subseteq A$. 
Let \( G = (A \cup B, E) \) be a bipartite graph which satisfies Hall’s condition. Define a poset \( P = A \cup B \) and define \(<\) by \( a < b \) only if \( a \in A, b \in B \) and \((a, b) \in E\).

Suppose that the largest anti-chain in \( P \) is \( A = \{a_1, a_2, \ldots, a_h, b_1, b_2, \ldots, b_k\} \) and let \( s = h + k \).

Now

\[
N(\{a_1, a_2, \ldots, a_h\}) \subseteq B \setminus \{b_1, b_2, \ldots, b_k\}
\]

for otherwise \( A \) will not be an anti-chain.

From Hall’s condition we see that

\[
|B| - k \geq h \text{ or equivalently } |B| \geq s.
\]
Now by Dilworth’s theorem, $P$ is the union of $s$ chains:

A matching $M$ of size $m$, $|A| - m$ members of $A$ and $|B| - m$ members of $B$.

But then

$$m + (|A| - m) + (|B| - m) = s \leq |B|$$

and so $m \geq |A|$.

□
A *network* consists of a *loopless* digraph $D = (V, A)$ plus a function $c : A \to \mathbb{R}_+$. Here $c(x, y)$ for $(x, y) \in A$ is the *capacity* of the edge $(x, y)$.

We use the following notation: if $\phi : A \to \mathbb{R}$ and $S, T$ are (not necessarily disjoint) subsets of $V$ then

$$\phi(S, T) = \sum_{x \in S, y \in T} \phi(x, y).$$

Let $s, t$ be distinct vertices. An $s - t$ flow is a function $f : A \to \mathbb{R}$ such that

$$f(v, V \setminus \{v\}) = f(V \setminus \{v\}, v) \quad \text{for all } v \neq s, t.$$ 

In words: flow into $v$ equals flow out of $v$. 

Covered so far
An $s - t$ flow is **feasible** if

$$0 \leq f(x, y) \leq c(x, y) \quad \text{for all } (x, y) \in A.$$ 

An $s - t$ **cut** is a partition of $V$ into two sets $S, \bar{S}$ such that $s \in S$ and $t \in \bar{S}$.

The **value** $v_f$ of the flow $f$ is given by

$$v_f = f(s, V \setminus \{s\}) - f(V \setminus \{s\}, s).$$

Thus $v_f$ is the net flow leaving $s$.

The **capacity** of the cut $S : \bar{S}$ is equal to $c(S, \bar{S})$. 
Max-Flow Min-Cut Theorem

Theorem

\[ \max v_f = \min c(S, \bar{S}) \]

where the maximum is over feasible s – t flows and the minimum is over s – t cuts.

Proof

We observe first that

\[ f(S, \bar{S}) - f(\bar{S}, S) = (f(S, V) - f(S, S)) - (f(V, S) - f(S, S)) \]
\[ = f(S, V) - f(V, S) \]
\[ = v_f + \sum_{v \in S \setminus \{s\}} (f(v, V) - f(V, v)) \]
\[ = v_f. \]

So,

\[ v_f \leq f(S, \bar{S}) \leq c(S, \bar{S}). \]
This implies that
\[
\max v_f \leq \min c(S, \bar{S}). \quad (11)
\]

Given a flow \( f \) we define a \textit{flow augmenting path} \( P \) to be a sequence of distinct vertices \( x_0 = s, x_1, x_2, \ldots, x_k = t \) such that for all \( i \), either

\begin{align*}
\text{F1} & \quad (x_i, x_{i+1}) \in A \text{ and } f(x_i, x_{i+1}) < c(x_i, x_{i+1}), \text{ or} \\
\text{F2} & \quad (x_{i+1}, x_i) \in A \text{ and } f(x_{i+1}, x_i) > 0.
\end{align*}

If \( P \) is such a sequence, then we define \( \theta_P > 0 \) to be the minimum over \( i \) of \( c(x_i, x_{i+1}) - f(x_i, x_{i+1}) \) (Case (F1)) and \( f(x_{i+1}, x_i) \) (Case (F2)).
Claim 1: $f$ is a maximum value flow, iff there are no flow augmenting paths.

Proof: If $P$ is flow augmenting then define a new flow $f'$ as follows:

1. $f'(x_i, x_{i+1}) = f(x_i, x_{i+1}) + \theta_P$ or
2. $f'(x_{i+1}, x_i) = f(x_{i+1}, x_i) - \theta_P$
3. For all other edges, $(x, y)$, we have $f'(x, y) = f(x, y)$.

We can see that the flow stays balanced at $x_i$. 

Covered so far
We can see then that if there is a flow augmenting path then the new flow satisfies

\[ v_{f'} = v_f + \theta_P > v_f. \]

Let \( S_f \) denote the set of vertices \( v \) for which there is a sequence \( x_0 = s, x_1, x_2, \ldots, x_k = v \) which satisfies F1, F2 of the definition of flow augmenting paths.

If \( t \in S_f \) then the associated sequence defines a flow augmenting path. So, assume that \( t \notin S_f \). Then we have,

1. \( s \in S_f \).
2. If \( x \in S_f, y \in \tilde{S}_f, (x, y) \in A \) then \( f(x, y) = c(x, y) \), else we would have \( y \in S_f \).
3. If \( x \in S_f, y \in \tilde{S}_f, (y, x) \in A \) then \( f(y, x) = 0 \), else we would have \( y \in S_f \).
We therefore have

\[ v_f = f(S_f, \overline{S_f}) - f(\overline{S_f}, S) = c(S, \overline{S_f}). \]

We see from this and (11) that \( f \) is a flow of maximum value and that the cut \( S_f \setminus \overline{S_f} \) is of minimum capacity.

This finishes the proof of Claim 1 and the Max-Flow Min-Cut theorem.

Note also that we can construct \( S_f \) by beginning with \( S_f = \{s\} \) and then repeatedly adding any vertex \( y \not\in S_f \) for which there is \( x \in S_f \) such that F1 or F2 holds. (A simple inductive argument based on sequence length shows that all of \( S_f \) is constructed in this way.)
Note also that we can construct $S_f$ by beginning with $S_f = \{s\}$ and then repeatedly adding any vertex $y \notin S_f$ for which there is $x \in S_f$ such that F1 or F2 holds.

This defines an algorithm for finding a maximum flow. The construction either finishes with $t \in S_f$ and we can augment the flow.

Or, we find that $t \notin S_f$ and we have a maximum flow.

Note, that if all the capacities $c(x, y)$ are integers and we start with the all zero flow then we find that $\theta_f$ is always a positive integer (formally one can use induction to verify this).

It follows that in this case, there is always a maximum flow that only takes integer values on the edges.
Hall’s Theorem.

Let $G = (A, B, E)$ be a bipartite graph with $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. A matching $M$ is a set of edges that meets each vertex at most once. A matching is perfect if it meets each vertex.

Hall’s theorem:

**Theorem**

$G$ contains a perfect matching iff $|N(S)| \geq |S|$ for all $S \subseteq A$.

Here $N(S) = \{b \in B : \exists a \in A \text{ s.t. } \{a, b\} \in E\}$.

Define a digraph $\Gamma$ by adding vertices $s, t \notin A \cup B$. Then add edges $(s, a_i)$ and $(b_i, t)$ of capacity 1 for $i = 1, 2, \ldots, n$. Orient the edges $E$ for $A$ to $B$ and give them capacity $\infty$. 
$G$ has a matching of size $m$ iff there is an $s - t$ flow of value $m$. An $s - t$ cut $X : X$ has capacity

$$|A \setminus X| + |B \cap X| + |\{a \in X \cap A, b \in B \setminus X : \{a, b\} \in E\}| \times \infty.$$ 

It follows that to find a minimum cut, we need only consider $X$ such that

$$\{a \in X \cap A, b \in B \setminus X : \{a, b\} \in E\} = \emptyset.$$ \hspace{1cm} (12)

For such a set, we let $S = A \cap X$ and $T = X \cap B$. Condition (12) means that $T \supseteq N(S)$. The capacity of $X : X$ is now $(n - |S|) + |T|$ and for a fixed $S$ this is minimised for $T = N(S)$.

Thus, by the Max-Flow Min-Cut theorem

$$\max\{|M|\} = \min_{X} \{c(X : \bar{X})\} = \min_{S} \{n - |S| + |N(S)|\}.$$ 

This implies Hall’s theorem.
Let $G = (V, E)$ be a graph. When is it possible to orient the edges of $G$ to create a digraph $\Gamma = (V, A)$ so that every vertex has out-degree at least $d$. We say that $G$ is $d$-orientable.

**Theorem**

$G$ is $d$-orientable iff

$$\left| \{ e \in E : e \cap S \neq \emptyset \} \right| \geq d|S| \text{ for all } S \subseteq V. \quad (13)$$

**Proof**

If $G$ is $d$-orientable then

$$\left| \{ e \in E : e \cap S \neq \emptyset \} \right| \geq \left| \{ (x, y) \in A : x \in S \} \right| \geq d|S|.$$
Suppose now that (13) holds. Define a network $D$ as follows; the vertices are $s, t, V, E$ – yes, $D$ has a vertex for each edge of $G$.

There is an edge of capacity $d$ from $s$ to each $v \in V$ and an edge of capacity one from each $e \in E$ to $t$. There is an edge of infinite capacity from $v \in V$ to each edge $e$ that contains $v$.

Consider an integer flow $f$. Suppose that $e = \{v, w\} \in E$ and $f(e, t) = 1$. Then either $f(v, e) = 1$ or $f(w, e) = 1$. In the former we interpret this as orienting the edge $e$ from $v$ to $w$ and in the latter from $w$ to $v$.

Under this interpretation, $G$ is $d$-orientable iff $D$ has a flow of value $d|V|$. 
Let $X : \overline{X}$ be an $s - t$ cut in $N$. Let $S = X \cap V$ and $T = X \cap E$.

To have a finite capacity, there must be no $x \in S$ and $e \in E \setminus T$ such that $x \in e$.

So, the capacity of a finite capacity cut is at least

$$d(|V| - |S|) + |\{ e \in E : e \cap S \neq \emptyset \}|$$

And this is at least $d|V|$ if (13) holds.
**Example 1** A disc lies in a plane. Its centre is fixed but it is free to rotate. It has been divided into $n$ sectors of angle $\frac{2\pi}{n}$. Each sector is to be colored Red or Blue. How many different colorings are there?

One could argue for $2^n$.

On the other hand, what if we only distinguish colorings which cannot be obtained from one another by a rotation. For example if $n = 4$ and the sectors are numbered 0,1,2,3 in clockwise order around the disc, then there are only 6 ways of coloring the disc – 4R, 4B, 3R1B, 1R3B, RRBB and RBRB.
Now consider an $n \times n$ “chessboard” where $n \geq 2$. Here we color the squares Red and Blue and two colorings are different only if one cannot be obtained from another by a rotation or a reflection. For $n = 2$ there are 6 colorings.
The general scenario that we consider is as follows: We have a set $X$ which will stand for the set of colorings when transformations are not allowed. (In example 1, $|X| = 2^n$ and in example 2, $|X| = 2^{n^2}$).

In addition there is a set $G$ of permutations of $X$. This set will have a group structure:

Given two members $g_1, g_2 \in G$ we can define their composition $g_1 \circ g_2$ by $g_1 \circ g_2(x) = g_1(g_2(x))$ for $x \in X$. We require that $G$ is closed under composition i.e. $g_1 \circ g_2 \in G$ if $g_1, g_2 \in G$. 
We also have the following:

A1 The *identity* permutation $1_X \in G$.

A2 $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ (Composition is associative).

A3 The inverse permutation $g^{-1} \in G$ for every $g \in G$.

(A set $G$ with a binary relation $\circ$ which satisfies A1, A2, A3 is called a Group).
In example 1, $D = \{0, 1, 2, \ldots, n - 1\}$, $X = 2^D$ and the group is $G_1 = \{e_0, e_1, \ldots, e_{n-1}\}$ where $e_j \ast x = x + j \mod n$ stands for rotation by $2j\pi/n$.

In example 2, $X = 2^{[n]^2}$. We number the squares 1,2,3,4 in clockwise order starting at the upper left and represent $X$ as a sequence from $\{r, b\}^4$ where for example rrbr means color 1,2,4 Red and 3 Blue. $G_2 = \{e, a, b, c, p, q, r, s\}$ is in a sense independent of $n$. $e, a, b, c$ represent a rotation through 0, 90, 180, 270 degrees respectively. $p, q$ represent reflections in the vertical and horizontal and $r, s$ represent reflections in the diagonals 1,3 and 2,4 respectively.
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10/20/2021
Theorem 2
(Frobenius, Burnside)

\[ \nu_{X,G} = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|. \]

Proof Let \( A(x, g) = 1_{g \ast x = x} \). Then

\[
\nu_{X,G} = \frac{1}{|G|} \sum_{x \in X} |S_x| \\
= \frac{1}{|G|} \sum_{x \in X} \sum_{g \in G} A(x, g) \\
= \frac{1}{|G|} \sum_{g \in G} \sum_{x \in X} A(x, g) \\
= \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|. 
\]
Let us consider example 1 with $n = 6$. We compute

<table>
<thead>
<tr>
<th>$g$</th>
<th>$e_0$</th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
<th>$e_4$</th>
<th>$e_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mid \text{Fix}(g) \mid$</td>
<td>64</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

Applying Theorem 2 we obtain

$$\nu_{X,G} = \frac{1}{6}(64 + 2 + 4 + 8 + 4 + 2) = 14.$$
Cycles of a permutation

Let $\pi : D \to D$ be a permutation of the finite set $D$. Consider the digraph $\Gamma_\pi = (D, A)$ where $A = \{(i, \pi(i)) : i \in D\}$. $\Gamma_\pi$ is a collection of vertex disjoint cycles. Each $x \in D$ being on a unique cycle. Here a cycle can consist of a loop i.e. when $\pi(x) = x$.


<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi(i)$</td>
<td>6</td>
<td>2</td>
<td>7</td>
<td>10</td>
<td>3</td>
<td>8</td>
<td>9</td>
<td>1</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

The cycles are $(1, 6, 8), (2), (3, 7, 9, 5), (4, 10)$. 
In general consider the sequence $i, \pi(i), \pi^2(i), \ldots$.

Since $D$ is finite, there exists a first pair $k < \ell$ such that $\pi^k(i) = \pi^\ell(i)$. Now we must have $k = 0$, since otherwise putting $x = \pi^{k-1}(i) \neq y = \pi^{\ell-1}(i)$ we see that $\pi(x) = \pi(y)$, contradicting the fact that $\pi$ is a permutation.

So $i$ lies on the cycle $C = (i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i), i)$.

If $j$ is not a vertex of $C$ then $\pi(j)$ is not on $C$ and so we can repeat the argument to show that the rest of $D$ is partitioned into cycles.
Example 2

It is straightforward to check that when \( n \) is even, we have

\[
\begin{array}{cccccccc}
  g & e & a & b & c & p & q & r & s \\
  \mid \text{Fix}(g) \mid & 2^n & 2^n/4 & 2^n/4 & 2^n/4 & 2^n/2 & 2^n/2 & 2^{n+1}/2 & 2^{n+1}/2 \\
\end{array}
\]

For example, if we divide the chessboard into \( 4 \times n/2 \times n/2 \) sub-squares, numbered 1,2,3,4 then a coloring is in \( \text{Fix}(a) \) iff each of these 4 sub-squares have colorings which are rotations of the coloring in square 1.
Polya’s Theorem

We now extend the above analysis to answer questions like: How many distinct ways are there to color an $8 \times 8$ chessboard with 32 white squares and 32 black squares?
The scenario now consists of a set $D$ (Domain, a set $C$ (colors) and $X = \{x : D \rightarrow C\}$ is the set of colorings of $D$ with the color set $C$. $G$ is now a group of permutations of $D$. 
We see first how to extend each permutation of \( D \) to a permutation of \( X \). Suppose that \( x \in X \) and \( g \in G \) then we define \( g \star x \) by

\[
g \star x(d) = x(g^{-1}(d)) \quad \text{for all } d \in D.
\]

**Explanation:** The color of \( d \) is the color of the element \( g^{-1}(d) \) which is mapped to it by \( g \).

Consider Example 1 with \( n = 4 \). Suppose that \( g = e_1 \) i.e. rotate clockwise by \( \pi/2 \) and \( x(1) = b, x(2) = b, x(3) = r, x(4) = r \). Then for example

\[
g \star x(1) = x(g^{-1}(1)) = x(4) = r, \text{ as before.}
\]
Now associate a weight $w_c$ with each $c \in C$. If $x \in X$ then

$$W(x) = \prod_{d \in D} w_{x(d)}.$$  

Thus, if in Example 1 we let $w(r) = R$ and $w(b) = B$ and take $x(1) = b, x(2) = b, x(3) = r, x(4) = r$ then we will write $W(x) = B^2R^2$.

For $S \subseteq X$ we define the inventory of $S$ to be

$$W(S) = \sum_{x \in S} W(x).$$

The problem we discuss now is to compute the pattern inventory $PI = W(S^*)$ where $S^*$ contains one member of each orbit of $X$ under $G$. 

Covered so far
For example, in the case of Example 2, with $n = 2$, we get


To see that the definition of $PI$ makes sense we need to prove

**Lemma 3** If $x, y$ are in the same orbit of $X$ then $W(x) = W(y)$.

**Proof** Suppose that $g \ast x = y$. Then

$$W(y) = \prod_{d \in D} w_y(d)$$

$$= \prod_{d \in D} w_{g \ast x}(d)$$

$$= \prod_{d \in D} w_x(g^{-1}(d))$$

$$= \prod_{d \in D} w_x(d)$$

$$= W(x)$$

Note, that we can go from (14) to (15) because as $d$ runs over $D$, $g^{-1}(d)$ also runs over $d$. 

Covered so far
Let $\Delta = |D|$. If $g \in G$ has $k_i$ cycles of length $i$ then we define

$$ct(g) = x_1^{k_1} x_2^{k_2} \cdots x_\Delta^{k_\Delta}.$$  

The **Cycle Index Polynomial** of $G$, $C_G$ is then defined to be

$$C_G(x_1, x_2, \ldots, x_\Delta) = \frac{1}{|G|} \sum_{g \in G} ct(g).$$

In Example 2 with $n = 2$ we have

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<tr>
<th>$g$</th>
<th>e</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>p</th>
<th>q</th>
<th>r</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ct(g)$</td>
<td>$x_1^4$</td>
<td>$x_4$</td>
<td>$x_2^2$</td>
<td>$x_4$</td>
<td>$x_2^2$</td>
<td>$x_2^2$</td>
<td>$x_1^2 x_2$</td>
<td>$x_1^2 x_2$</td>
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</table>

and so

$$C_G(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 3x_2^2 + 2x_1^2 x_2 + 2x_4).$$
In Example 2 with $n = 3$ we have

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<tr>
<th>g</th>
<th>e</th>
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<th>b</th>
<th>c</th>
<th>p</th>
<th>q</th>
<th>r</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ct(g)$</td>
<td>$x_1^9$</td>
<td>$x_1 x_4^2$</td>
<td>$x_1 x_2^4$</td>
<td>$x_1 x_4^2$</td>
<td>$x_1^3 x_2^3$</td>
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<td>$x_1 x_2^4$</td>
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and so

$$C_G(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^9 + x_1 x_2^4 + 4x_1^3 x_2^3 + 2x_1 x_4^2).$$
Theorem (Polya)

\[ PI = C_G \left( \sum_{c \in C} w_c, \sum_{c \in C} w_c^2, \ldots, \sum_{c \in C} w_c^\Delta \right). \]

Proof In Example 2, we replace \( x_1 \) by \( R + B \), \( x_2 \) by \( R^2 + B^2 \) and so on. When \( n = 2 \) this gives

\[ PI = \frac{1}{8} \left( (R + B)^4 + 3(R^2 + B^2)^2 + 2(R + B)^2(R^2 + B^2) + 2(R^4 + B^4) \right) \]
\[ = R^4 + R^3B + 2R^2B^2 + RB^3 + B^4. \]

Putting \( R = B = 1 \) gives the number of distinct colorings. Note also the formula for \( PI \) tells us that there are 2 distinct colorings using 2 reds and 2 Blues.
Proof of Polya’s Theorem

Let \( X = X_1 \cup X_2 \cup \cdots \cup X_m \) be the equivalence classes of \( X \) under the relation

\[
x \sim y \text{ iff } W(x) = W(y).
\]

By Lemma 2, \( g \ast x \sim x \) for all \( x \in X, g \in G \) and so we can think of \( G \) acting on each \( X_i \) individually i.e. we use the fact that \( x \in X_i \) implies \( g \ast x \in X_i \) for all \( i \in [m], g \in G \). We use the notation \( g^{(i)} \in G^{(i)} \) when we restrict attention to \( X_i \).
Let $m_i$ denote the number of orbits $\nu_{X_{i},G^{(i)}}$ and $W_i$ denote the common PI of $G^{(i)}$ acting on $X_i$. Then

\[ PI = \sum_{i=1}^{m} m_i W_i \]

\[ = \sum_{i=1}^{m} W_i \left( \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g^{(i)})| \right) \]  

by Theorem 2

\[ = \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{m} |\text{Fix}(g^{(i)})| W_i \]

\[ = \frac{1}{|G|} \sum_{g \in G} W(\text{Fix}(g)) \]  

(16)

Note that (16) follows from $\text{Fix}(g) = \bigcup_{i=1}^{m} \text{Fix}(g^{(i)})$ since $x \in \text{Fix}(g^{(i)})$ iff $x \in X_i$ and $g \ast x = x$. 

Covered so far
Suppose now that $ct(g) = x_1^{k_1} x_2^{k_2} \cdots x_\Delta^{k_\Delta}$ as above. Then we claim that

$$W(Fix(g)) = \left( \sum_{c \in C} w_c \right)^{k_1} \left( \sum_{c \in C} w_c^2 \right)^{k_2} \cdots \left( \sum_{c \in C} w_c^\Delta \right)^{k_\Delta}. \quad (17)$$

Substituting (17) into (16) yields the theorem.

To verify (17) we use the fact that if $x \in Fix(g)$, then the elements of a cycle of $g$ must be given the same color. A cycle of length $i$ will then contribute a factor $\sum_{c \in C} w_c^i$ where the term $w_c^i$ comes from the choice of color $c$ for every element of the cycle. \hfill $\Box$
10/27/2021
**Game 1** Start with \( n \) chips. Players A,B alternately take 1,2,3 or 4 chips until there are none left. The winner is the person who takes the last chip:

**Example**

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<tr>
<th>( n = 10 )</th>
<th>A</th>
<th>B</th>
<th>A</th>
<th>B</th>
<th>A</th>
<th>B wins</th>
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<tr>
<td>( n = 11 )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>A wins</td>
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</table>

What is the optimal strategy for playing this game?
**Game 2** Chip placed at point \((m, n)\). Players can move chip to \((m', n)\) or \((m, n')\) where \(0 \leq m' < m\) and \(0 \leq n' < n\). The player who makes the last move and puts the chip onto \((0, 0)\) wins.

What is the optimal strategy for this game?

**Game 2a** Chip placed at point \((m, n)\). Players can move chip to \((m', n)\) or \((m, n')\) or to \((m - a, n - a)\) where \(0 \leq m' < m\) and \(0 \leq n' < n\) and \(0 \leq a \leq \min\{m, n\}\). The player who makes the last move and puts the chip onto \((0, 0)\) wins.

What is the optimal strategy for this game?
Game 3  $W$ is a set of words. A and B alternately remove words $w_1, w_2, \ldots$, from $W$. The rule is that the first letter of $w_{i+1}$ must be the same as the last letter of $w_i$. The player who makes the last legal move wins.

Example

$W = \{ \text{England, France, Germany, Russia, Bulgaria, ...} \}$

What is the optimal strategy for this game?
Abstraction

Represent each position of the game by a vertex of a digraph $D = (X, A)$. 

$(x, y)$ is an arc of $D$ iff one can move from position $x$ to position $y$.

We assume that the digraph is finite and that it is acyclic i.e. there are no directed cycles.

The game starts with a token on vertex $x_0$ say, and players alternately move the token to $x_1, x_2, \ldots$, where $x_{i+1} \in N^+(x_i)$, the set of out-neighbours of $x_i$. The game ends when the token is on a sink i.e. a vertex of out-degree zero. The last player to move is the winner.
Example 1: \( V(D) = \{0, 1, \ldots, n\} \) and \((x, y) \in A\) iff \(x - y \in \{1, 2, 3, 4\}\).

Example 2: \( V(D) = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\} \) and \((x, y) \in N^+((x', y'))\) iff \(x = x'\) and \(y > y'\) or \(x > x'\) and \(y = y'\).

Example 2a: \( V(D) = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\} \) and \((x, y) \in N^+((x', y'))\) iff \(x = x'\) and \(y > y'\) or \(x > x'\) and \(y = y'\) or \(x - x' = y - y' > 0\).

Example 3: \( V(D) = \{(W', w) : W' \subseteq W \setminus \{w\}\}. \) \(w\) is the last word used and \(W'\) is the remaining set of unused words. \((X', w') \in N^+((X, w))\) iff \(w' \in X\) and \(w'\) begins with the last letter of \(w\). Also, there is an arc from \((W, \cdot)\) to \((W \setminus \{w\}, w)\) for all \(w\), corresponding to the games start.
We will first argue that such a game must eventually end.

A **topological numbering** of digraph $D = (X, A)$ is a map $f : X \to [n]$, $n = |X|$ which satisfies $(x, y) \in A$ implies $f(x) < f(y)$.

**Theorem**

A finite digraph $D = (X, A)$ is acyclic iff it admits at least one topological numbering.

**Proof** Suppose first that $D$ has a topological numbering. We show that it is acyclic.

Suppose that $C = (x_1, x_2, \ldots, x_k, x_1)$ is a directed cycle. Then $f(x_1) < f(x_2) < \cdots < f(x_k) < f(x_1)$, contradiction.
Abstraction

Suppose now that $D$ is acyclic. We first argue that $D$ has at least one sink.

Thus let $P = (x_1, x_2, \ldots, x_k)$ be a longest simple path in $D$. We claim that $x_k$ is a sink.

If $D$ contains an arc $(x_k, y)$ then either $y = x_i, 1 \leq i \leq k - 1$ and this means that $D$ contains the cycle $(x_i, x_{i+1}, \ldots, x_k, x_i)$, contradiction or $y \notin \{x_1, x_2, \ldots, x_k\}$ and then $(P, y)$ is a longer simple path than $P$, contradiction.
We can now prove by induction on \( n \) that there is at least one topological numbering.

If \( n = 1 \) and \( X = \{x\} \) then \( f(x) = 1 \) defines a topological numbering.

Now assume that \( n > 1 \). Let \( z \) be a sink of \( D \) and define \( f(z) = n \). The digraph \( D' = D - z \) is acyclic and by the induction hypothesis it admits a topological numbering, \( f : X \setminus \{z\} \to [n - 1] \).

The function we have defined on \( X \) is a topological numbering. If \( (x, y) \in A \) then either \( x, y \neq z \) and then \( f(x) < f(y) \) by our assumption on \( f \), or \( y = z \) and then \( f(x) < n = f(z) \) (\( x \neq z \) because \( z \) is a sink).
The fact that $D$ has a topological numbering implies that the game must end. Each move increases the $f$ value of the current position by at least one and so after at most $n$ moves a sink must be reached.

The positions of a game are partitioned into 2 sets:
- $P$-positions: The next player cannot win. The previous player can win regardless of the current player’s strategy.
- $N$-positions: The next player has a strategy for winning the game.

Thus an $N$-position is a winning position for the next player and a $P$-position is a losing position for the next player.

The main problem is to determine $N$ and $P$ and what the strategy is for winning from an $N$-position.
Abstraction

Let the vertices of $D$ be $x_1, x_2, \ldots, x_n$, in topological order.

**Labelling procedure**

1. $i \leftarrow n$, Label $x_n$ with $P$. $N \leftarrow \emptyset$, $P \leftarrow \emptyset$.
2. $i \leftarrow i - 1$. If $i = 0$ STOP.
3. Label $x_i$ with $N$, if $N^+(x_i) \cap P \neq \emptyset$.
4. Label $x_i$ with $P$, if $N^+(x_i) \subseteq N$.
5. goto 2.

The partition $N, P$ satisfies

$$x \in N \iff N^+(x) \cap P \neq \emptyset$$

To play from $x \in N$, move to $y \in N^+(x) \cap P$. 
In Game 1, $P = \{5k : k \geq 0\}$.

In Game 2, $P = \{(x, x) : x \geq 0\}$.

**Lemma**

*The partition into $N, P$ satisfying $x \in N$ iff $N^+(x) \cap P \neq \emptyset$ is unique.*

**Proof** If there were two partitions $N_i, P_i$, $i = 1, 2$, let $x_i$ be the vertex of highest topological number which is not in $(N_1 \cap N_2) \cup (P_1 \cap P_2)$. Suppose that $x_i \in N_1 \setminus N_2$.

But then $x_i \in N_1$ implies $N^+(x_i) \cap P_1 \cap \{x_{i+1}, \ldots, x_n\} \neq \emptyset$ and $x_i \in P_2$ implies $N^+(x_i) \cap P_2 \cap \{x_{i+1}, \ldots, x_n\} = \emptyset$.

But $P_1 \cap \{x_{i+1}, \ldots, x_n\} = P_2 \cap \{x_{i+1}, \ldots, x_n\}$. 

Covered so far
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Sums of games

Suppose that we have \( p \) games \( G_1, G_2, \ldots, G_p \) with digraphs \( D_i = (X_i, A_i), \ i = 1, 2, \ldots, p. \) The sum \( G_1 \oplus G_2 \oplus \cdots \oplus G_p \) of these games is played as follows. A position is a vector \( (x_1, x_2, \ldots, x_p) \in X = X_1 \times X_2 \times \cdots \times X_p. \) To make a move, a player chooses \( i \) such that \( x_i \) is not a sink of \( D_i \) and then replaces \( x_i \) by \( y \in N_i^+(x_i). \) The game ends when each \( x_i \) is a sink of \( D_i \) for \( i = 1, 2, \ldots, n. \)

Knowing the partitions \( N_i, P_i \) for game \( i = 1, 2, \ldots, p \) does not seem to be enough to determine how to play the sum of the games.

We need more information. This will be provided by the Sprague-Grundy Numbering.
Example

Nim In a one pile game, we start with $a \geq 0$ chips and while there is a positive number $x$ of chips, a move consists of deleting $y \leq x$ chips. In this game the $N$-positions are the positive integers and the unique $P$-position is 0.

In general, Nim consists of the sum of $n$ single pile games starting with $a_1, a_2, \ldots, a_n > 0$. A move consists of deleting some chips from a non-empty pile.

Example 2 is Nim with 2 piles.
Sprague-Grundy (SG) Numbering

For $S \subseteq \{0, 1, 2, \ldots, \}$ let

$$\text{mex}(S) = \min \{x \geq 0 : x \notin S\}.$$ 

Now given an acyclic digraph $D = X, A$ with topological ordering $x_1, x_2, \ldots, x_n$ define $g$ iteratively by

1. $i \leftarrow n$, $g(x_n) = 0$.
2. $i \leftarrow i - 1$. If $i = 0$ STOP.
3. $g(x_i) = \text{mex}(\{g(x) : x \in N^+(x_i)\})$.
4. goto 2.
Lemma

\[ x \in P \iff g(x) = 0. \]

Proof

Because

\[ x \in N \iff N^+(x) \cap P \neq \emptyset \]

all we have to show is that

\[ g(x) > 0 \iff \exists y \in N^+(y) \text{ such that } g(y) = 0. \]

But this is immediate from \[ g(x) = \text{mex} \left( \{ g(y) : y \in N^+(x) \} \right) \]
Another one pile subtraction game.

- A player can remove any even number of chips, but not the whole pile.
- A player can remove the whole pile if it is odd.

The terminal positions are 0 or 2.

Lemma

\[ g(0) = 0, \quad g(2k) = k - 1 \quad \text{and} \quad g(2k - 1) = k \quad \text{for} \quad k \geq 1. \]
Proof 0,2 are terminal positions and so \( g(0) = g(2) = 0 \).
\( g(1) = 1 \) because the only position one can move to from 1 is 0. We prove the remainder by induction on \( k \).

Assume that \( k > 1 \).

\[
\begin{align*}
g(2k) &= \text{mex}\{g(2k - 2), g(2k - 4), \ldots, g(2)\} \\
     &= \text{mex}\{k - 2, k - 3, \ldots, 0\} \\
     &= k - 1.
\end{align*}
\]

\[
\begin{align*}
g(2k - 1) &= \text{mex}\{g(2k - 3), g(2k - 5), \ldots, g(1), g(0)\} \\
         &= \text{mex}\{k - 1, k - 2, \ldots, 0\} \\
         &= k.
\end{align*}
\]
We now show how to compute the $SG$ numbering for a sum of games.

For binary integers $a = a_m a_{m-1} \cdots a_1 a_0$ and $b = b_m b_{m-1} \cdots b_1 b_0$ we define $a \oplus b = c_m c_{m-1} \cdots c_1 c_0$ by

$$c_i = \begin{cases} 
1 & \text{if } a_i \neq b_i \\
0 & \text{if } a_i = b_i 
\end{cases}$$

for $i = 1, 2, \ldots, m$.

So $11 \oplus 5 = 14$. 
Sums of games

Theorem

If $g_i$ is the SG function for game $G_i$, $i = 1, 2, \ldots, p$ then the SG function $g$ for the sum of the games $G = G_1 \oplus G_2 \oplus \cdots \oplus G_p$ is defined by

$$g(x) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_p(x_p)$$

where $x = (x_1, x_2, \ldots, x_p)$.

For example if in a game of Nim, the pile sizes are $x_1, x_2, \ldots, x_p$ then the SG value of the position is

$$x_1 \oplus x_2 \oplus \cdots \oplus x_p$$
Sums of games

**Proof** It is enough to show this for \( p = 2 \) and then use induction on \( p \).

Write \( G = H \oplus G_p \) where \( H = G_1 \oplus G_2 \oplus \cdots \oplus G_{p-1} \). Let \( h \) be the \( SG \) numbering for \( H \). Then, if \( y = (x_1, x_2, \ldots, x_{p-1}) \),

\[
g(x) = h(y) \oplus g_p(x_p) \quad \text{assuming theorem for } p = 2
= g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_{p-1}(x_{p-1}) \oplus g_p(x_p)
\]

by induction.

It is enough now to show, for \( p = 2 \), that

- **A1** If \( x \in X \) and \( g(x) = b > a \) then there exists \( x' \in N^+(x) \) such that \( g(x') = a \).
- **A2** If \( x \in X \) and \( g(x) = b \) and \( x' \in N^+(x) \) then \( g(x') \neq g(x) \).
- **A3** If \( x \in X \) and \( g(x) = 0 \) and \( x' \in N^+(x) \) then \( g(x') \neq 0 \)
Sums of games

A1. Write \( d = a \oplus b \). Then
\[
a = d \oplus b = d \oplus g_1(x_1) \oplus g_2(x_2).
\]
(18)

Now suppose that we can show that either

\((i) \ d \oplus g_1(x_1) < g_1(x_1) \) or \( (ii) \ d \oplus g_2(x_2) < g_2(x_2) \) or both. \( (19) \)

Assume that \((i)\) holds.

Then since \( g_1(x_1) = \text{mex}(N_1^+(x_1)) \) there must exist \( x'_1 \in N_1^+(x_1) \) such that \( g_1(x'_1) = d \oplus g_1(x_1) \).

Then from (18) we have
\[
a = g_1(x'_1) \oplus g_2(x_2) = g(x'_1, x_2).
\]

Furthermore, \( (x'_1, x_2) \in N^+(x) \) and so we will have verified A1.
Let us verify (19).

Suppose that $2^{k-1} \leq d < 2^k$.

Then $d$ has a 1 in position $k$ and no higher.

Since $d_k = a_k \oplus b_k$ and $a < b$ we must have $a_k = 0$ and $b_k = 1$.

So either (i) $g_1(x_1)$ has a 1 in position $k$ or (ii) $g_2(x_2)$ has a 1 in position $k$. Assume (i).

But then $d \oplus g_1(x_1) < g_1(x_1)$ since $d$ “destroys” the $k$th bit of $g_1(x_1)$ and does not change any higher bit.
A2. Suppose without loss of generality that $g(x_1', x_2) = g(x_1, x_2)$ where $x_1' \in N^+(x_1)$.

Then $g_1(x_1') \oplus g_2(x_2) = g_1(x_1) \oplus g_2(x_2)$ implies that $g_1(x_1') = g_1(x_1)$, contradiction. □

A3. Suppose that $g_1(x_1) \oplus g_2(x_2) = 0$ and $g_1(x_1') \oplus g_2(x_2) = 0$ where $x_1' \in N^+(x_1)$.

Then $g_1(x_1) = g_1(x_1')$, contradicting $g_1(x_1) = mex\{g_1(x) : x \in N^+(x_1)\}$. 
If we apply this theorem to the game of Nim then if the position $x$ consists of piles of $x_i$ chips for $i = 1, 2, \ldots, p$ then 

$$g(x) = x_1 \oplus x_2 \oplus \cdots \oplus x_p.$$ 

In our first example, $g(x) = x \mod 5$ and so for the sum of $p$ such games we have 

$$g(x_1, x_2, \ldots, x_p) = (x_1 \mod 5) \oplus (x_2 \mod 5) \oplus \cdots \oplus (x_p \mod 5).$$
Geography

Start with a chip sitting on a vertex \( v \) of a graph or digraph \( G \). A move consists of moving the chip to a neighbouring vertex.

In edge geography, moving the chip from \( x \) to \( y \) deletes the edge \((x, y)\). In vertex geography, moving the chip from \( x \) to \( y \) deletes the vertex \( x \).

The problem is given a position \((G, v)\), to determine whether this is a \( P \) or \( N \) position.

**Complexity** Both edge and vertex geography are Pspace-hard on digraphs. Edge geography is Pspace-hard on an undirected graph. Only vertex geography on a graph is polynomial time solvable.
We need some simple results from the theory of matchings on graphs. A *matching* $M$ of a graph $G = (V, E)$ is a set of edges, no two of which are incident to a common vertex.
An $M$-alternating path joining 2 $M$-unsaturated vertices is called an $M$-augmenting path.
$M$ is a \textit{maximum} matching of $G$ if no matching $M'$ has more edges.

**Theorem**

$M$ is a maximum matching iff $M$ admits no $M$-augmenting paths.

**Proof**  
Suppose $M$ has an augmenting path $P = (a_0, b_1, a_1, \ldots, a_k, b_{k+1})$ where  
$e_i = (a_{i-1}, b_i) \notin M$, $1 \leq i \leq k + 1$ and  
$f_i = (b_i, a_i) \in M$, $1 \leq i \leq k$.

Let $M' = M - \{f_1, f_2, \ldots, f_k\} + \{e_1, e_2, \ldots, e_{k+1}\}$.
Undirected Vertex Geography

- $|M'| = |M| + 1$.
- $M'$ is a matching

For $x \in V$ let $d_M(x)$ denote the degree of $x$ in matching $M$, So $d_M(x)$ is 0 or 1.

$$d_{M'}(x) = \begin{cases} 
  d_M(x) & x \not\in \{a_0, b_1, \ldots, b_{k+1}\} \\
  d_M(x) & x \in \{b_1, \ldots, a_k\} \\
  d_M(x) + 1 & x \in \{a_0, b_{k+1}\}
\end{cases}$$

So if $M$ has an augmenting path it is not maximum.
Suppose $M$ is not a maximum matching and $|M'| > |M|$. Consider $H = G[M \nabla M']$ where $M \nabla M' = (M \setminus M') \cup (M' \setminus M)$ is the set of edges in *exactly* one of $M, M'$. Maximum degree of $H$ is $2 - \leq 1$ edge from $M$ or $M'$. So $H$ is a collection of vertex disjoint alternating paths and cycles.

$|M'| > |M|$ implies that there is at least one path of type (d). Such a path is $M$-augmenting
Theorem

\((G, v)\) is an \(N\)-position in UVG iff every maximum matching of \(G\) covers \(v\).

Proof

(i) Suppose that \(M\) is a maximum matching of \(G\) which covers \(v\). Player 1’s strategy is now: Move along the \(M\)-edge that contains the current vertex.

If Player 1 were to lose, then there would exist a sequence of edges \(e_1, f_1, \ldots, e_k, f_k\) such that \(v \in e_1, e_1, e_2, \ldots, e_k \in M, f_1, f_2, \ldots, f_k \notin M\) and \(f_k = (x, y)\) where \(y\) is the current vertex for Player 1 and \(y\) is not covered by \(M\).

But then if \(A = \{e_1, e_2, \ldots, e_k\}\) and \(B = \{f_1, f_2, \ldots, f_k\}\) then \((M \setminus A) \cup B\) is a maximum matching (same size as \(M\)) which does not cover \(v\), contradiction.
(ii) Suppose now that there is some maximum matching \( M \) which does not cover \( v \). If \((v, w)\) is Player 1’s move, then \( w \) must be covered by \( M \), else \( M \) is not a maximum matching.

Player 2’s strategy is now: Move along the \( M \)-edge that contains the current vertex. If Player 2 were to lose then there exists \( e_1 = (v, w), f_1, \ldots, e_k, f_k, e_{k+1} = (x, y) \) where \( y \) is the current vertex for Player 2 and \( y \) is not covered by \( M \).

But then we have defined an augmenting path from \( v \) to \( y \) and so \( M \) is not a maximum matching, contradiction. □
Note that we can determine whether or not \( v \) is covered by all maximum matchings as follows: Find the size \( \sigma \) of the maximum matching \( G \).

This can be done in \( O(n^3) \) time on an \( n \)-vertex graph. Find the size \( \sigma' \) of a maximum matching in \( G - v \). Then \( v \) is covered by all maximum matchings of \( G \) iff \( \sigma \neq \sigma' \).
We consider the following multi-dimensional version of Tic Tac Toe (Noughts and Crosses to the English).

The *board* consists of $[n]^d$. A point on the board is therefore a vector $(x_1, x_2, \ldots, x_d)$ where $1 \leq x_i \leq n$ for $1 \leq i \leq d$.

A *line* is a set points $(x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(d)})$, $j = 1, 2, \ldots, n$ where each sequence $x^{(i)}$ is either (i) of the form $k, k, \ldots, k$ for some $k \in [n]$ or is (ii) $1, 2, \ldots, n$ or is (iii) $n, n-1, \ldots, 1$. Finally, we cannot have Case (i) for all $i$.

Thus in the (familiar) $3 \times 3$ case, the top row is defined by $x^{(1)} = 1, 1, 1$ and $x^{(2)} = 1, 2, 3$ and the diagonal from the bottom left to the top right is defined by $x^{(1)} = 3, 2, 1$ and $x^{(2)} = 1, 2, 3$. 
Lemma

The number of winning lines in the \((n, d)\) game is \(\frac{(n+2)^d - n^d}{2}\).

Proof

In the definition of a line there are \(n\) choices for \(k\) in (i) and then (ii), (iii) make it up to \(n + 2\). There are \(d\) independent choices for each \(i\) making \((n + 2)^d\).

Now delete \(n^d\) choices where only Case (i) is used. Then divide by 2 because replacing (ii) by (iii) and vice-versa whenever Case (i) does not hold produces the same set of points (traversing the line in the other direction). □
The game is played by 2 players. The Red player (X player) goes first and colours a point red. Then the Blue player (0 player) colours a different point blue and so on.

A player wins if there is a line, all of whose points are that players colour. If neither player wins then the game is a draw. The second player does not have a winning strategy:

Lemma

*Player 1 can always get at least a draw.*
Proof  We prove this by considering *strategy stealing*.

Suppose that Player 2 did have a winning strategy. Then Player 1 can make an arbitrary first move $x_1$. Player 2 will then move with $y_1$. Player 1 will now win playing the winning strategy for Player 2 against a first move of $y_1$.

This can be carried out until the strategy calls for move $x_1$ (if at all). But then Player 1 can make an arbitrary move and continue, since $x_1$ has already been made.

□

The Hales-Jewett Theorem of Ramsey Theory implies that there is a winner in the $(n, d)$ game, when $n$ is large enough with respect to $d$. The winner is of course Player 1.
11/3/2021
The above array gives a strategy for Player 2 in the $5 \times 5$ game ($d = 2, n = 5$).

For each of the 12 lines there is an associated pair of positions. If Player 1 chooses a position with a number $i$, then Player 2 responds by choosing the other cell with the number $i$.

This ensures that Player 1 cannot take line $i$. If Player 1 chooses the * then Player 2 can choose any cell with an unused number.
So, later in the game if Player 1 chooses a cell with $j$ and Player 2 already has the other $j$, then Player 2 can choose an arbitrary cell.

Player 2’s strategy is to ensure that after all cells have been chosen, he/she will have chosen one of the numbered cells associated with each line. This prevents Player 1 from taking a whole line. This is called a *pairing* strategy.
We now generalise the game to the following: We have a family $\mathcal{F} = A_1, A_2, \ldots, A_N \subseteq A$. A move consists of one player, taking an uncoloured member of $A$ and giving it his colour.

A player wins if one of the sets $A_i$ is completely coloured with his colour.

A pairing strategy is a collection of distinct elements $X = \{x_1, x_2, \ldots, x_{2N-1}, x_{2N}\}$ such that $x_{2i-1}, x_{2i} \in A_i$ for $i \geq 1$.

This is called a *draw forcing pairing*. Player 2 responds to Player 1’s choice of $x_{2i+\delta}, \delta = 0, 1$ by choosing $x_{2i+3-\delta}$. If Player 1 does not choose from $X$, then Player 2 can choose any uncoloured element of $X$. 
In this way, Player 2 avoids defeat, because at the end of the game Player 2 will have coloured at least one of each of the pairs $x_{2i-1}$, $x_{2i}$ and so Player 1 cannot have completely coloured $A_i$ for $i = 1, 2, \ldots, N$. 
Theorem

If
\[
\left| \bigcup_{A \in \mathcal{G}} A \right| \geq 2|\mathcal{G}| \quad \forall \mathcal{G} \subseteq \mathcal{F}
\] (20)

then there is a draw forcing pairing.

Proof  We define a bipartite graph $\Gamma$. $A$ will be one side of the bipartition and $B = \{b_1, b_2, \ldots, b_{2N}\}$. Here $b_{2i-1}$ and $b_{2i}$ both represent $A_i$ in the sense that if $a \in A_i$ then there is an edge $(a, b_{2i-1})$ and an edge $(a, b_{2i})$.

A draw forcing pairing corresponds to a complete matching of $B$ into $A$ and the condition (20) implies that Hall’s condition is satisfied. □
Corollary

If $|A_i| \geq n$ for $i = 1, 2, \ldots, n$ and every $x \in A$ is contained in at most $n/2$ sets of $\mathcal{F}$ then there is a draw forcing pairing.

Proof  The degree of $a \in A$ is at most $2(n/2)$ in $\Gamma$ and the degree of each $b \in B$ is at least $n$. This implies (via Hall’s condition) that there is a complete matching of $B$ into $A$. \hfill \Box
Consider Tic tac Toe when $d = 2$. If $n$ is even then every array element is in at most 3 lines (one row, one column and at most one diagonal) and if $n$ is odd then every array element is in at most 4 lines (one row, one column and at most two diagonals).

Thus there is a draw forcing pairing if $n \geq 6$, $n$ even and if $n \geq 9$, $n$ odd. (The cases $n = 4, 7$ have been settled as draws. $n = 7$ required the use of a computer to examine all possible strategies.)
In general we have

**Lemma**

If \( n \geq 3^d - 1 \) and \( n \) is odd or if \( n \geq 2^d - 1 \) and \( n \) is even, then there is a draw forcing pairing of \((n, d)\) Tic tac Toe.

**Proof** We only have to estimate the number of lines through a fixed point \( \mathbf{c} = (c_1, c_2, \ldots, c_d) \).

If \( n \) is odd then to choose a line \( L \) through \( \mathbf{c} \) we specify, for each index \( i \) whether \( L \) is (i) constant on \( i \), (ii) increasing on \( i \) or (iii) decreasing on \( i \).

This gives \( 3^d \) choices. Subtract 1 to avoid the all constant case and divide by 2 because each line gets counted twice this way.
When $n$ is even, we observe that once we have chosen in which positions $L$ is constant, $L$ is determined.

Suppose $c_1 = x$ and 1 is not a fixed position. Then every other non-fixed position is $x$ or $n - x + 1$. Assuming w.l.o.g. that $x \leq n/2$ we see that $x < n - x + 1$ and the positions with $x$ increase together at the same time as the positions with $n - x + 1$ decrease together.

Thus the number of lines through $c$ in this case is bounded by $\sum_{i=0}^{d-1} \binom{d}{i} = 2^d - 1$. \qed
We now prove a theorem of Erdős and Selfridge.

**Theorem**

If $|A_i| \geq n$ for $i \in [N]$ and $N < 2^{n-1}$, then Player 2 can get a draw in the game defined by $\mathcal{F}$.

**Proof** At any point in the game, let $C_j$ denote the set of elements in $A$ which have been coloured with Player $j$’s colour, $j = 1, 2$ and $U = A \setminus C_1 \cup C_2$. Let

$$\Phi = \sum_{i : A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|}.$$

Suppose that the players choices are $x_1, y_1, x_2, y_2, \ldots$. Then we observe that immediately after Player 1’s first move, $\Phi < N2^{-(n-1)} < 1$. 

Covered so far
We will show that Player 2 can keep $\Phi < 1$ throughout. Then at the end, when $U = \emptyset$, $\Phi = \sum_{i: A_i \cap C_2 = \emptyset} 1 < 1$ implies that $A_i \cap C_2 \neq \emptyset$ for all $i \in [N]$.

So, now let $\Phi_j$ be the value of $\Phi$ after the choice of $x_1, y_1, \ldots, x_j$. then if $U, C_1, C_2$ are defined at precisely this time,

$$
\Phi_{j+1} - \Phi_j = - \sum_{i: A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|} + \sum_{i: A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|}
$$

$$
\leq - \sum_{i: A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|} + \sum_{i: A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|}
$$

Covered so far
We deduce that $\Phi_{j+1} - \Phi_j \leq 0$ if Player 2 chooses $y_j$ to maximise $\sum_{i \colon A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|}$ over $y$.

In this way, Player 2 keeps $\Phi < 1$ and obtains a draw.

In the case of $(n, d)$ Tic Tac Toe, we see that Player 2 can force a draw if

$$\frac{(n + 2)^d - n^d}{2} < 2^{n-1}$$

which is implied, for $n$ large, by

$$n \geq (1 + \varepsilon)d \log_2 d$$

where $\varepsilon > 0$ is a small positive constant.