Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2020: Test 4

Name: ________________________________

Andrew ID: ________________________________

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Q1: (25pts)
Let $G = (V, E)$ be a graph with $m$ edges. Let $\mathcal{C}$ denote the set of cycles of $G$. For $C \in \mathcal{C}$ we let $|C|$ denote the number of edges in $C$. Prove that

$$\sum_{C \in \mathcal{C}} \frac{1}{m|C|} \leq 1.$$

Solution: The set $\mathcal{C}$ is a Sperner family. If $C_1, C_2 \in \mathcal{C}$ then $C_1 \nsubseteq C_2$. The inequality follows directly from the LYM inequality.
Q2: (25pts)

Let \( E = \{e_1, \ldots, e_m\} \) and suppose that \( S_j \subseteq E, j = 1, \ldots, n \) contains \( s_j \) elements. Suppose also that \( e_i, i = 1, \ldots, m \) occurs in \( r_i \) of the sets \( S_j \). Let \( S = \sum_{j=1}^n s_j = \sum_{i=1}^m r_i \) and \( M = \max\{r_1, \ldots, r_m, s_1, \ldots, s_n\} \). Show that if \( s > (n-1)M \) then there exist distinct \( e_t, t = 1, \ldots, n \) such that \( e_t \in S_t, t = 1, \ldots, n \).

(Hint: Hall’s theorem)

**Solution:** consider the bipartite graph \( \Gamma \) with vertices \( A = \{a_1, a_2, \ldots, a_n\} \) and \( B = \{b_1, b_2, \ldots, b_m\} \) where edge \((a_i, b_j)\) exists iff \( e_j \in S_i \). We have to prove that \( \Gamma \) contains a matching of \( A \) into \( B \). If there is no such matching, then there exists \( X \subseteq A \) such that \( |N(X)| < |X| \).

\[
\ell = \left| \bigcup_{i \in X} S_i \right| < k = |X|.
\]

This implies that there is an element \( e \in E \) that occurs in at least \( p \) of the sets \( S_i, i \in X \) where \( p\ell \geq \sum_{i \in X} s_j \). By assumption we have

\[
\sum_{i \in X} s_i + \sum_{i \not\in X} s_i > (n-1)M.
\]

Thus

\[
p\ell > (n-1)M - \sum_{i \not\in X} s_i \geq (n-1)M - (n-k)M = (k-1)M.
\]

Because \( \ell \leq k-1 \), this implies that \( p > M \), a contradiction.
Q3: (25pts)
Find the set of $P$-positions for the take-away games with subtraction sets

(a) $S = \{1, 3, 7\}$.

(b) $S = \{1, 4, 6\}$.

(Reminder: in a take-away game with subtraction set $S$, a player can only remove $x$ from a pile, if $x \in S$.)

Suppose now that there are two piles and the rules for each pile are as above. Now find the $P$ positions for the two pile game where in one pile $S$ is as in (a) and the other pile is as in (b).

**Solution:** let $g_a, g_b$ denote the SG-numbers for the two games. We have

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An easy induction shows that

$$g_a(n) = n \mod 2 \text{ and } g_b(n) = \begin{cases} 0 & n = 0, 2 \mod 5. \\ 1 & n = 1, 3 \mod 5. \\ 2 & n = 4 \mod 5. \end{cases}$$

The $P$-position for the two-pile game are when $g_a(bn \oplus g_b(n)) = 0$ or

$$P = \{n : n \mod 10 \in \{0, 1, 2, 3, 4\}\}.$$
Q4: (25pts)
How many ways are there to arrange 4C’s, 4 G’s, 5 A’s and 8T’s under the condition that any arrangement and its reverse/inverse are to be considered the same.

Solution: The group $G$ consists of $\{e, a\}$ where $a$ is a reflection through the middle of the word. Now

$$|\text{Fix}(e)| = \frac{21!}{4!4!5!8!},$$
$$|\text{Fix}(a)| = \frac{10!}{2!2!2!4!}.$$

A sequence is in $\text{Fix}(a)$ if it is a palindrome i.e. looks the same backwards as forwards. It must have middle letter A. Then we arrange 2 C’s, 2 G’s, 2 A”s and 4 T’s in any order and then complete the sequence uniquely to a palindrome.

The total number of arrangements is $(|\text{Fix}(e)| + |\text{Fix}(a)|)/2.$