21-301 Combinatorics
Homework 6
Due: Monday, October 26

1. Suppose that in a town of n citizens, clubs $C_1, C_2, \ldots, C_m$ must satisfy (i) $|C_i|$ is even for $i = 1, 2, \ldots, m$ and (ii) $|C_i \cap C_j|$ is odd for $1 \leq i < j \leq m$. Show that $m \leq n$.
(Hint: Consider the cases n odd and n even separately. In the n even case consider the matrix $M = AA^T$ where A is made up from the incidence vectors of the columns of $C_1, C_2, \ldots, C_m$.)

2. Let $A_1, A_2, \ldots, A_m \subseteq [n]$ be such that for $1 \leq i < j \leq m$, $d_{i,j} = |(A_i \setminus A_j) \cup (A_j \setminus A_i)|$ takes one of two values. Show that $m \leq (n+1)(n+4)/2$.

3. Each edge of K_n appears an odd number of times as an edge in the collection $G_1, G_2, \ldots, G_m$ of bipartite subgraphs of K_n. Show that $m \geq (n-1)/2$.
[Hint: Let A_k, B_k, M_k, S be as in the notes on Linear Algebraic Methods and consider the $2n \times n$ matrix $T = \begin{bmatrix} S \\ S^T \\ \mathbf{1}^T \end{bmatrix}$.]