21-301 Combinatorics
Homework 4
Due: Wednesday, October 7

1. Subsets $A_i, B_i \subseteq [n]$, $i = 1, 2, \ldots, m$ satisfy $A_i \cap B_i = \emptyset$ for all i and $A_i \cap B_j \neq \emptyset$ for all $i \neq j$. Show that
$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$
Solution: Let π be a random permutation of $[n]$ and for disjoint sets A, B define the event $\mathcal{E}(A, B)$ by
$$\mathcal{E}(A, B) = \{\pi : \max\{\pi(a) : a \in A\} < \min\{\pi(b) : b \in B\}\}.$$
The events $\mathcal{E}_i = \mathcal{E}(A_i, B_i)$, $i = 1, 2, \ldots, m$ are disjoint. Indeed, suppose that $\mathcal{E}(A_i, B_i)$ and $\mathcal{E}(A_j, B_j)$ occur. Let $x \in A_i \cap B_j$ and $y \in A_j \cap B_i$. x, y exist by (ii) and (i) implies that they are distinct. Then $\mathcal{E}(A_i, B_i)$ implies that $\pi(x) < \pi(y)$ and $\mathcal{E}(A_j, B_j)$ implies that $\pi(x) > \pi(y)$, contradiction.
Observe next that for two fixed disjoint sets A, B, $|A| = a, |B| = b$ there are exactly $\binom{n}{a+b} a!b!(n-a-b)!$ permutations that produce the event $\mathcal{E}(A, B)$. Indeed, there are $\binom{n}{a+b}$ places to position $A \cup B$. Then there are $a!b!$ that place A as the first a of these $a+b$ places. Finally, there are $(n-a-b)!$ ways of ordering the remaining elements not in $A \cup B$.
Thus
$$\begin{aligned}\Pr(\mathcal{E}(A_i, B_i)) &= \frac{n!}{(|A_i| + |B_i|)!(n - |A_i| - |B_i|)!} |A_i|!|B_i|!(n - |A_i| - |B_i|)! \frac{1}{n!} \\ &= = \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}}.\end{aligned}$$
But then the disjointness of the collection of events $\mathcal{E}(A_i, B_i)$ implies that
$$\sum_{i=1}^{m} \Pr(\mathcal{E}(A_i, B_i)) \leq 1.$$

2. Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be distinct finite subsets of $\{1, 2, 3, \ldots, \}$ such that
   - for every i, $A_i \cap B_i = \emptyset$, and
   - for every $i \neq j$, $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$.

   Prove that for every real number $0 \leq p \leq 1$.
$$\sum_{i=1}^{n} p^{|A_i|}(1-p)^{|B_i|} \leq 1. \tag{1}$$
(Hint: Define disjoint events $\mathcal{E}_i$ such that the LHS of (1) is $\sum_i \Pr(\mathcal{E}_i)$.)

Solution: Choose a random subset X by including each integer with probability p. Then let

$$\mathcal{E} = \{A_i \subseteq X \text{ and } B_i \cap X = \emptyset\}.$$

Then

$$\Pr(\mathcal{E}_i) = p^{|A_i|}(1-p)^{|B_i|}$$

and $\mathcal{E}_i, \mathcal{E}_j$ are disjoint for $i \neq j$ since for example $A_i \subseteq X$ and $A_i \cap B_j \neq \emptyset$ implies that $B_j \cap X \neq \emptyset$.

3. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_i \geq 1$ for $i = 1, 2, \ldots, n$. Let J be any open interval of width 2. Show that of the 2^n sums $\sum_{i=1}^n \epsilon_i x_i$, ($\epsilon_i = \pm 1$), at most $\binom{n}{\lfloor n/2 \rfloor}$ lie in J.
   (Hint: use Sperner's lemma.)

   Solution: For $A \subseteq [n]$ let $x_A = \sum_{i \in A} x_i - \sum_{i \notin A} x_i$. Let $\mathcal{A} = \{A : x_A \in J\}$. It is enough to show that $\mathcal{A}$ is a Sperner family. Indeed, if $A, B \in \mathcal{A}$ and $A \subset B$ then $x_B - x_A = 2\sum_{i \in B \setminus A} x_i \geq 2$. Thus we cannot have both $x_A, x_B \in J$.