21-301 Combinatorics
Homework 4
Due: Wednesday, October 7

1. Subsets $A_i, B_i \subseteq [n]$, $i = 1, 2, \ldots, m$ satisfy $A_i \cap B_i = \emptyset$ for all i and $A_i \cap B_j \neq \emptyset$ for all $i \neq j$. Show that
$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

2. Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be distinct finite subsets of $\{1, 2, 3, \ldots,\}$ such that
   - for every i, $A_i \cap B_i = \emptyset$, and
   - for every $i \neq j$, $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$.

   Prove that for every real number $0 \leq p \leq 1$.
$$\sum_{i=1}^{n} p^{|A_i|}(1-p)^{|B_i|} \leq 1. \tag{1}$$
   (Hint: Define disjoint events $\mathcal{E}_i$ such that the LHS of (1) is $\sum_i \Pr(\mathcal{E}_i)$.)

3. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_i \geq 1$ for $i = 1, 2, \ldots, n$. Let J be any open interval of width 2. Show that of the 2^n sums $\sum_{i=1}^{n} \epsilon_i x_i$, ($\epsilon_i = \pm 1$), at most $\binom{n}{\lfloor n/2 \rfloor}$ lie in J.
   (Hint: use Sperner's lemma.)