9\9\15

Probabilistic Method

Coloring problem: $A_1, A_2, \cdots, A_n$ are subsets of A, $|A_i| = k$, $i = 1, 2, \cdots, n$.

If $n < 2^{k-1}$ then there exists a partition of $A = \underset{\text{red}}{R} \cup \underset{\text{blue}}{B}$

such that

$$A_i \cap R \neq \emptyset \;\&\; A_i = B \neq \emptyset$$

for $i = 1, 2, \cdots, n$

Note: if n is too large then it may not be possible to find such a coloring.

Suppose $n = \binom{2k}{k}$ and
$\{A_1, A_2, \ldots, A_n\} = \binom{[2k]}{k}$.

In any "coloring" there is a set ~~R~~ R or B of size $\geq k$. Suppose if R is a then any k-subset of R is red.

The ~~...~~ largest possible value of n (in terms of k) is not

How can we find a "good" coloring?

We choose R, B uniformly at random and show that with positive probability we get a "good" coloring.

$$BAD(i) = \{A_i \subseteq R \text{ or } A_i \subseteq B\}.$$

$$BAD = \bigcup_{i=1}^{n} BAD(i)$$

Want to show that

$$P(BAD) < 1$$

$\Longrightarrow$ $\exists$ a coloring with desired property.

If there were no coloring then we would have $P(BAD) = 1$

$$P(BAD) = P\left[\bigcup_{i=1}^{n} BAD(i)\right]$$

$$\leq \sum_{i=1}^{n} Pr[BAD(i)]$$

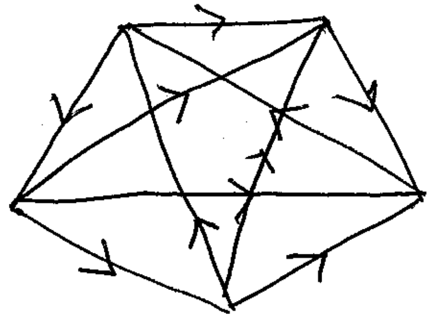
$$= \sum_{i=1}^{n} \left(\frac{1}{2}\right)^{k-1}$$

$$= \frac{n}{2^{k-1}} < 1.$$

$\square$

Tournaments

A tournament is an orientation of a complete graph K_n.



For $1 \le i < j \le n$ there is an edge (i,j) or (j,i) but not both

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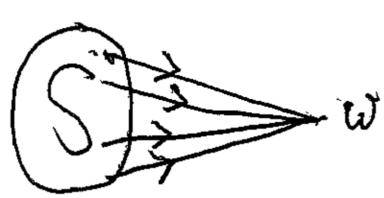
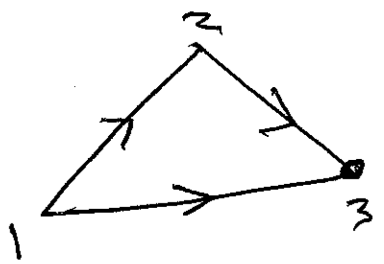
Let $\mathcal{A}_k$ be the property that for every set $S \subseteq [n]$, $|S| = k$ there exists $w \notin S$ such that w "beats" S i.e. all edges are oriented from S to w.



Not every tournament will have property $\mathcal{A}_1$, say.



$S = \{3\}$ breaks $\mathcal{A}_1$.

But there exist tournaments with $\mathcal{A}_1$

<u>Theorem</u>

For every k there exists a tournament with property $\mathcal{A}_k$.

To prove there exist a tournament with property $\mathcal{A}_k$, we choose n large and choose our tournament T at random i.e. we independently orient the edges in either direction.

Bad events: ~~...~~ for $|S| = k$ we let

$$\mathcal{B}_S = \{ \nexists\, w \text{ that beats } S \}.$$

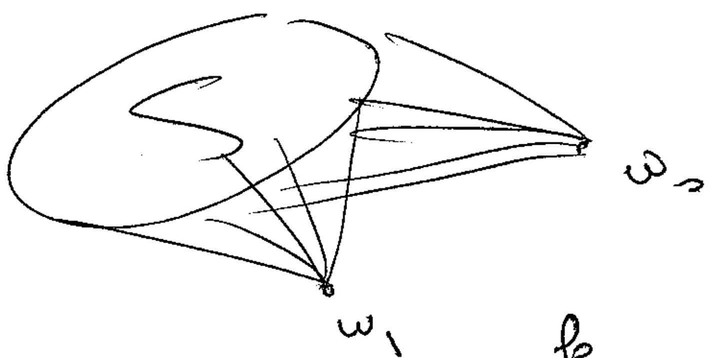
$$\overline{\mathcal{A}_k} = \bigcup_{|S|=k} \mathcal{B}_S$$

$$\Pr(\bar{\mathcal{A}}_k) \leq \sum_{|S|=k} \Pr(\mathcal{B}_S)$$

$$\mathcal{B}_S = \bigcap_{w \notin S} \mathcal{B}_{S,w}$$

where $\mathcal{B}_{S,w} = \{ w \text{ does not beat } S \}$

The events $\left.\mathcal{B}_{S,w}\right\}_{w \notin S}$ are independent for a fixed S.



$\mathcal{B}_{S,w_1}$ & $\mathcal{B}_{S,w_2}$ depend on disjoint "coin flips".

$$\Pr(\neg \mathcal{B}_{S,w}) = \left(\frac{1}{2}\right)^k$$

$$\Pr(\mathcal{B}_{S,w}) = 1 - \left(\frac{1}{2}\right)^k$$

$$\Pr(\mathcal{B}_S) = \left(1 - \left(\frac{1}{2}\right)^k\right)^{n-k}$$

$$
\begin{aligned}
\Pr(\neg \mathcal{A}_k) &\leq \sum_S \Pr(\mathcal{B}_S) \\
&\leq \sum_S \left(1 - \frac{1}{2^k}\right)^{n-k} \\
&\leq \binom{n}{k}\left(1 - \frac{1}{2^k}\right)^{n-k} \\
&< 1 \quad \text{if } n \gg k.
\end{aligned}
$$