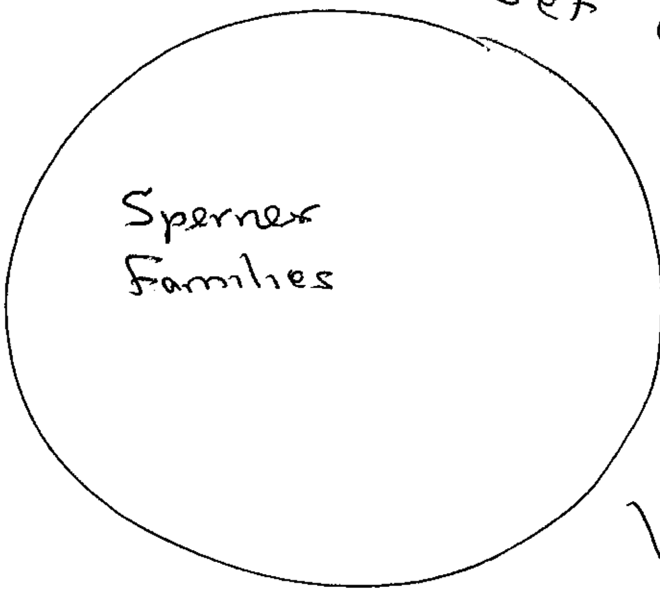
9/30/15

Some extremal problems:



Set of combinatorial objects.

Each has a "size".

What can you say about the size of the largest (smallest) object.

Sperner Family

$\mathcal{P}_n = 2^{[n]}$ = power set. $\mathcal{A} \subseteq \mathcal{P}_n$ is "Sperner Family" if $A, B \in \mathcal{A} \Rightarrow A \not\subseteq B$ & $B \not\subseteq A$.

?? $|\mathcal{A}| \leq$??

If we take all sets of size $\lfloor n/2 \rfloor$ then we get a family of size $\binom{n}{\lfloor n/2 \rfloor}$.

Can we do better?

<u>Theorem</u>

If $\mathcal{A} \subseteq \mathcal{P}_n$ is a Sperner family then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

<u>Proof</u>

We will show that

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1 \qquad \text{LYM inequality}$$

$$1 \geq \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{\lfloor \frac{1}{2} n \rfloor}} = \frac{|\mathcal{A}|}{\binom{n}{\lfloor n/2 \rfloor}}.$$

Let π be a random permutation of $[n]$.

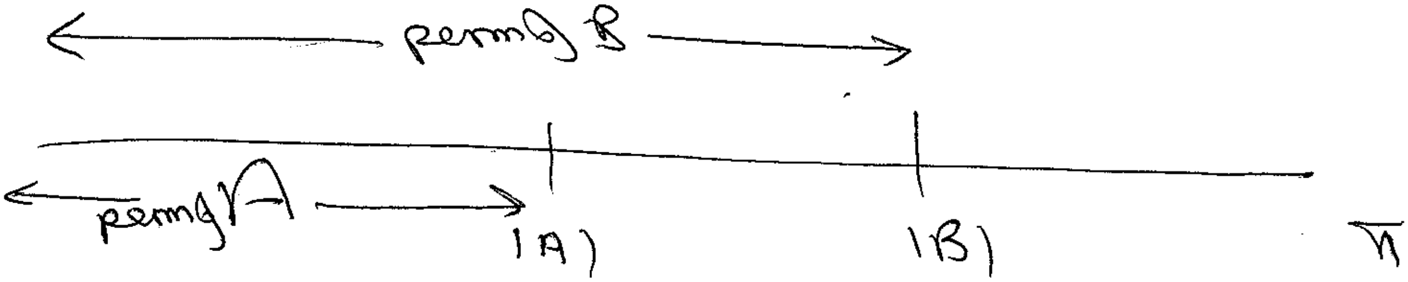
For $A \in \mathcal{A}$ let $\mathcal{E}_A$ be the event

$$\{\pi(1), \pi(2), \dots, \pi(|A|)\} = A$$

First $|A|$ elements of π consist of A.

$n = 6$: $A = \{1, 5\}$ $\quad \mathcal{E}_A =$ $1, 5, \dots$ or $5, 1, \dots$

Key observation $A, B \in \mathcal{A} \Rightarrow \mathcal{E}_A \cap \mathcal{E}_B = \emptyset$.



$\Rightarrow B \supseteq A$ — not possible.

$$\sum_{A \in \mathcal{A}} P_r(\mathcal{E}_A) \le 1$$

$= \frac{1}{\binom{n}{|A|}}$ because first $|A|$ elements of π are random

Suppose instead we have integer $s \geq 1$ and a family $\mathcal{A}$ such that there do not exist distinct $A_1, A_2, \ldots, A_{s+1} \in \mathcal{A}$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{s+1}$.

Theorem

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq s$$

Proof

π is a random permutation of $[n]$ and $\mathcal{E}_A$ is as before.

Z = # of events that occur.

$Z \leq s$ by assumption.

$$E(Z) = \sum_{A \in \mathcal{A}} \Pr(\mathcal{E}_A) = \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}}$$

Intersecting Families

(i) Suppose $\mathcal{A}$ is such that $A, B \in \mathcal{A}$
$\Rightarrow A \cap B = \emptyset$. $\mathcal{A} \subseteq \mathcal{P}_n$

$|\mathcal{A}| \leq n+1$.

(ii) Suppose $\mathcal{A}$ is such that $A, B \in \mathcal{A}$
implies $A \cap B \neq \emptyset$

Ex: $\mathcal{A} = \{A : 1 \in A\}$
$|\mathcal{A}| = 2^{n-1}$.

Claim: $|\mathcal{A}| \leq 2^{n-1}$.

Divide $\mathcal{P}_n$ into 2^{n-1} pairs $(S, \overline{S})$.

If $|\mathcal{A}| > 2^{n-1}$ then pigeon-hole-principle
$\Rightarrow \mathcal{A}$ contains a set S & its complement.