<u>9/28/15</u>

k-regular digraph G i.e. indegree = outdegree $= k$ for every vertex.

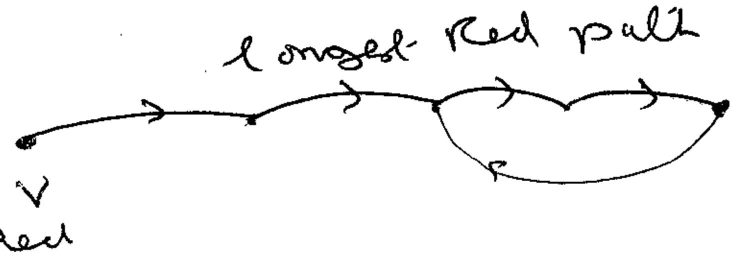
Claim: G contains $r = \left\lfloor \frac{k}{4\log k} \right\rfloor$ vertex disjoint cycles.

<u>Proof</u>

Color vertices of G <u>randomly</u> with colors $[r]$.

~~$\mathcal{E}_v$ = { v has a neighbor of the same color }~~

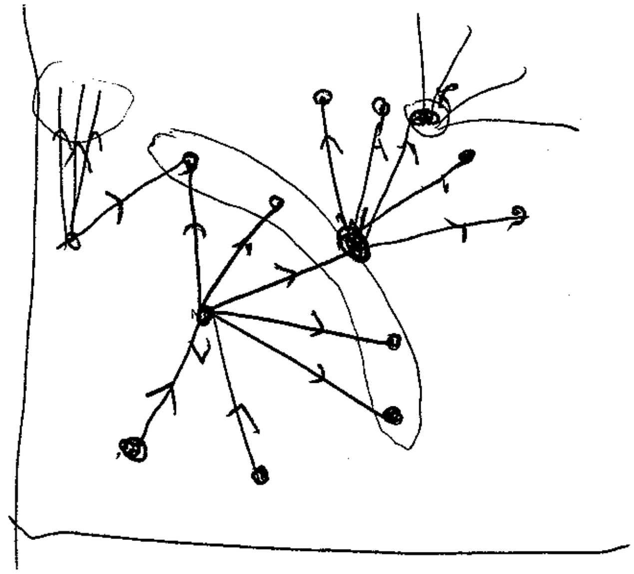
We prove $\bigcap_v \mathcal{E}_v \neq \emptyset$ ie $\Pr(\cdot) > 0$.



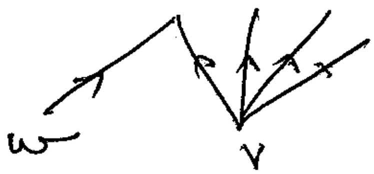
$\mathcal{E}_v$ = { vertex "sees" a neighbor of every color }.

$$\Pr[\neg \mathcal{E}_v] \leq r\left(1 - \frac{1}{r}\right)^k \leq r e^{-k/r} \leq r e^{-4\log k} \leq k^{-3}.$$

Dependency Graph



$\mathcal{E}_v$ is dependent on



w such that $N^+(w) \cap N^+(v) \neq \emptyset$

max degree in dependency graph $d \leq k^2$

Need $4dp \leq 1$ $\qquad 4k^2 \times k^{-3} = \frac{4}{k} \leq 1$.

Proof of symmetric local lemma

ie $4dp \le 1 \Rightarrow P\left(\bigcap_{i=1}^{m} \bar{\mathcal{E}}_i\right) \ge (1-2p)^m > 0$

We prove by induction on $|S|$ that for any $i \notin S$

$$(I_S) \quad \boxed{P\left(\mathcal{E}_i \,\middle|\, \bigcap_{j \in S} \bar{\mathcal{E}}_j\right) \le 2p}$$

This suffices because

$$P\left(\bigcap_{i=1}^{m} \bar{\mathcal{E}}_i\right) = \prod_{i=1}^{m} P\left(\bar{\mathcal{E}}_i \,\middle|\, \bigcap_{j=1}^{i-1} \bar{\mathcal{E}}_j\right)$$

$\cancel{P(A|BC)}$

$$P(ABC) = \frac{P(ABC)}{P(BC)} \cdot \frac{P(BC)}{P(C)} \cdot P(C)$$

$$= P(A|BC) \times P(B|C) \times P(C)$$

Base Case: $|S| = 0$

$$P(\mathcal{E}_i) \leq \cancel{2}p \qquad \checkmark$$

<u>Inductive Step</u>

Assume $i = n$, $S = \{1, 2, \ldots, s\}$,

$(i, x) \notin \Gamma$ for $x > d$.

$$P\left(\mathcal{E}_n \,\middle|\, \bigcap_{i=1}^{s} \overline{\mathcal{E}}_i\right) = \frac{P\left(\mathcal{E}_n \cap \bigcap_{i=1}^{d} \overline{\mathcal{E}}_i \,\middle|\, \bigcap_{i=d+1}^{s} \overline{\mathcal{E}}_i\right)}{P\left(\bigcap_{i=1}^{d} \overline{\mathcal{E}}_i \,\middle|\, \bigcap_{i=d+1}^{s} \overline{\mathcal{E}}_i\right)} \quad *$$

$$\leq \frac{P\left(\mathcal{E}_n \,\middle|\, \bigcap_{i=d+1}^{s} \overline{\mathcal{E}}_i\right)}{P\left(\bigcap_{i=1}^{d} \overline{\mathcal{E}}_i \,\middle|\, \bigcap_{i=d+1}^{s} \overline{\mathcal{E}}_i\right)}$$

$$* \quad \frac{P\left(\mathcal{E}_n \cap \bigcap_{i=1}^{d} \overline{\mathcal{E}}_i \cap \bigcap_{i=d+1}^{s} \overline{\mathcal{E}}_i\right) \Big/ P\left(\bigcap_{i=d+1}^{s} \overline{\mathcal{E}}_i\right)}{P\left(\bigcap_{i=1}^{d} \overline{\mathcal{E}}_i \cap \bigcap_{i=d+1}^{s} \overline{\mathcal{E}}_i\right) \Big/ P\left(\bigcap_{i=d+1}^{s} \overline{\mathcal{E}}_i\right)}$$

$$\leqslant \frac{P\left(\mathcal{E}_n \,\middle|\, \bigcap_{i=d+1}^{s} \bar{\mathcal{E}}_i\right)}{\Pr\left(\bigcap_{i=1}^{d} \bar{\mathcal{E}}_i \,\middle|\, \bigcap_{i=d+1}^{s} \bar{\mathcal{E}}_i\right)}$$

$$= \frac{P(\mathcal{E}_n)}{P\left(\bigcap_{i=1}^{d} \bar{\mathcal{E}}_i \,\middle|\, \bigcap_{i=d+1}^{s} \bar{\mathcal{E}}_i\right)}$$

DeMorgan Rule

$$= \frac{P(\mathcal{E}_n)}{1 - P\left(\bigcup_{i=1}^{d} \bar{\mathcal{E}}_i \,\middle|\, \bigcap_{i=d+1}^{s} \bar{\mathcal{E}}_i\right)}$$

Can assume $d > 0$, else events are independent

$$\leqslant \frac{P(\mathcal{E}_n)}{1 - \underbrace{\sum_{i=1}^{d} P\left(\bar{\mathcal{E}}_i \,\middle|\, \bigcap_{i=d+1}^{s} \bar{\mathcal{E}}_i\right)}_{\leq 2p \text{ by induction}}}$$

$$\leqslant \frac{p}{1-2dp} \leqslant \frac{p}{1-\frac{1}{2}} = 2p. \qquad \underline{4dp \leq 1}$$