9/18/15

Chernoff-Hoeffding Bounds

$S_n = X_1 + X_2 + \cdots + X_n$ [X_i are independent

$0 \leq X_i \leq 1$

$i = 1, 2, \ldots, n$

$E(X_i) = \mu_i$

We want to estimate

$\Pr[S_n \geq \mu + t]$

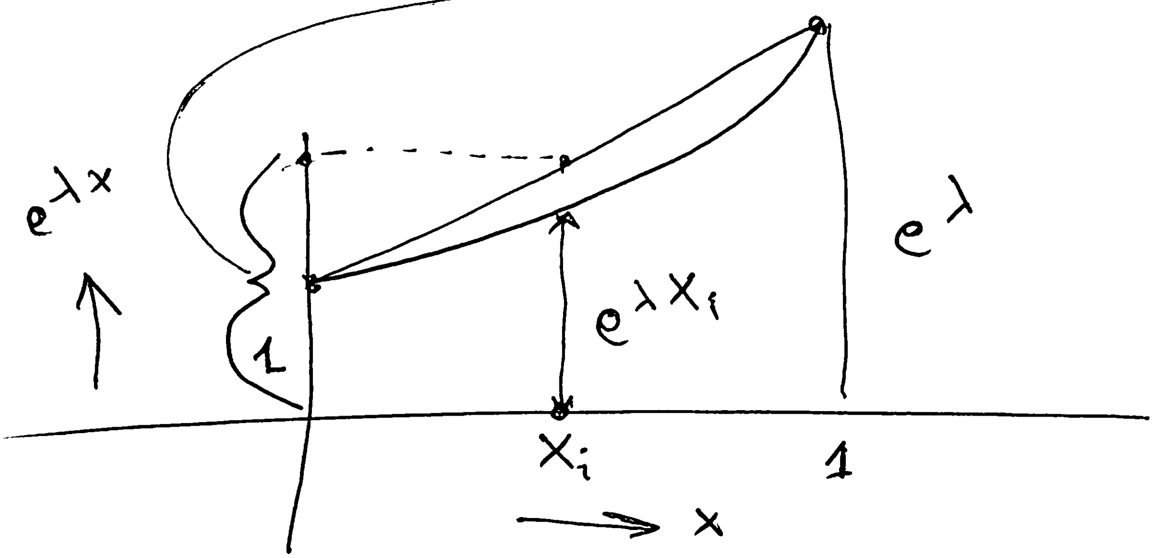
where $\mu = \mu_1 + \mu_2 + \cdots + \mu_n$.

$$P(S_n \geq \mu + t) \leq e^{-\lambda(\mu+t)} E(e^{\lambda S_n})$$

$$= e^{-\lambda(\mu+t)} \prod_{i=1}^{n} E(e^{\lambda X_i}).$$

Convexity implies that

$$e^{\lambda X_i} \leq 1 - X_i + X_i e^{\lambda}$$



$$E(e^{\lambda X_i}) \leq 1 - \mu_i + \mu_i e^{\lambda}$$

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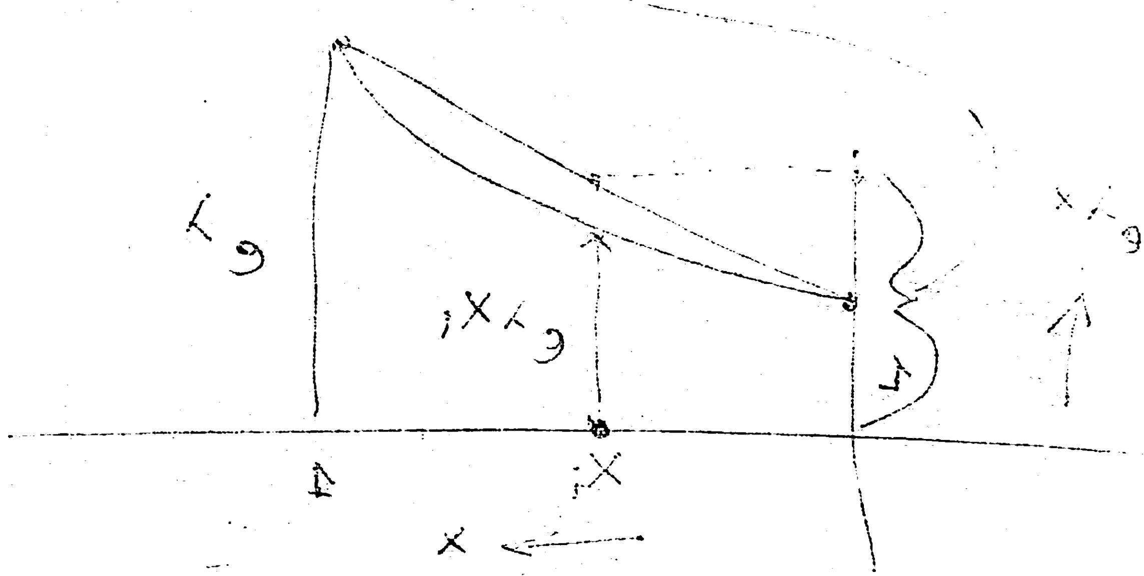

$$E(e^{\lambda X_i}) \leq 1 - \mu_i + \mu_i e^{\lambda}$$

$$P(S_n \geq \mu + t) \leq e^{-\lambda(\mu+t)} \prod_{i=1}^{n} (1 - \mu_i + \mu_i e^{\lambda})$$

$$\leq e^{-\lambda(\mu+t)} \left(\frac{n - \mu + \mu e^{\lambda}}{n} \right)^n$$

Geometric mean $\leq$ arithmetic mean.

$$(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Choose λ – calculus

$$e^{\lambda} = \frac{(\mu+t)(n-\mu)}{(n-\mu-t)\mu}$$

minimizes

⋮

$$P\left(\Sigma_n \geq \mu + a\right) \leq e^{-\lambda(\mu+a)} \prod_{i=1}^{n} \left(1 - p_i + p_i e^{\lambda}\right)$$

$$\leq e^{-\lambda(\mu+a)} \left(\frac{n - \mu + \mu e^{\lambda}}{n} \right)^{n}$$

Geometric mean $\leq$ arithmetic mean

$$\left(x_1 x_2 \cdots x_n\right)^{1/n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Choose λ – calculus

$$e^{\lambda} = \frac{(\mu + a)(n - \mu)}{\mu(n - \mu - a)}$$

minimize

$$P(S_n \geq \mu + t) \leq \exp\left\{ -\frac{t^2}{2(\mu + t/3)} \right\}$$

$$P(S_n \leq \mu - t) \leq \exp\left\{ -\frac{t^2}{2(\mu - \frac{t}{3})} \right\}$$

Remember these: $t = \epsilon\mu$

$$P(S_n \geq (1+\epsilon)\mu) \leq e^{-\frac{\mu\epsilon^2}{3}}$$

$$P(S_n \leq (1-\epsilon)\mu) \leq e^{-\frac{\mu\epsilon^2}{2}}$$

$$Pr(S_n \geq c\mu) \leq \left(\frac{e}{c}\right)^{c\mu}$$

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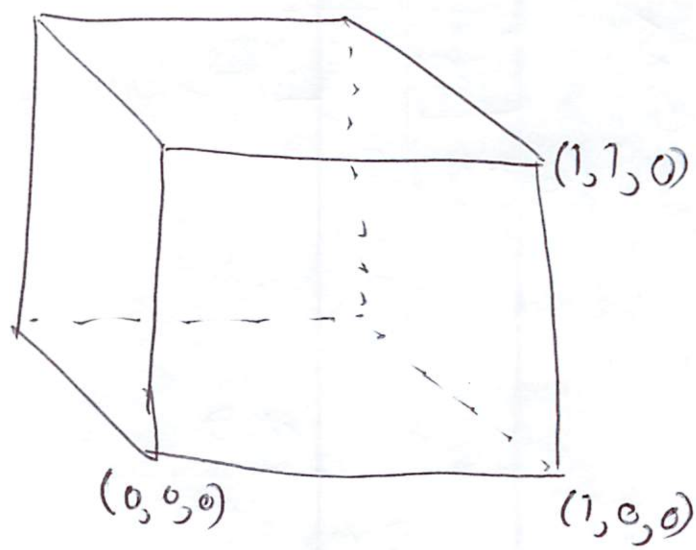
Valiant-Brebner Routing Algorithm

Q_n = n-cube

$V_n = \{0,1\}^n$



Each $x \in V_n$ wants to send a packet from $x \to \pi(x)$ where π is a permutation.

Problem is that one one packet can cross an edge at one time.

Send the packets sychronously – each time step on any ≤1 packet crosses

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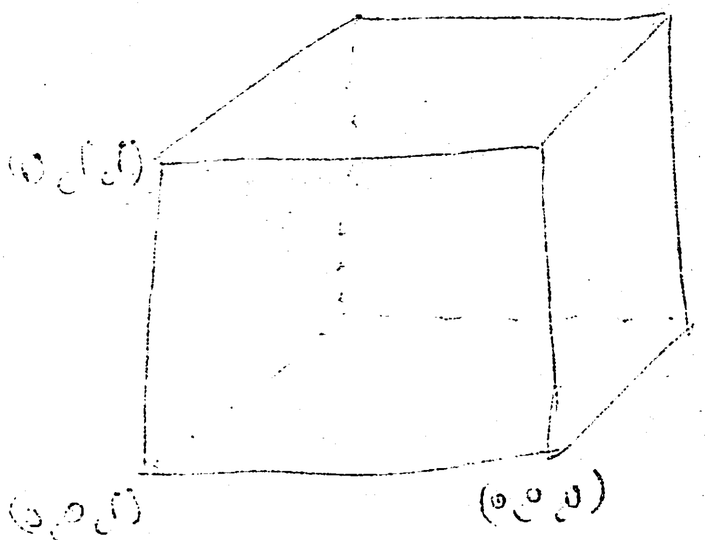

Each $x \in V_n$ wants to send a packet from $x \longrightarrow \pi(x)$ where π is a permutation.

Problem: that only one packet can cross an edge at one time.

Send the packets synchronously — each time step on any ≤ 1 packet crosses

We need a randomised algorithm — it is known that for any deterministic routing algorithm, there will be bad permutations that take a long time to deliver.

<u>Bit Fixing Path</u> $BFP(x, y)$

$$x = x_1 \; x_2 \; x_3 \cdots x_n$$
$$\downarrow$$
$$y_1 \; x_2 \; x_3 \cdots x_n$$
$$\downarrow$$
$$y_1 \; y_2 \; x_3 \cdots x_n$$
$$\vdots$$
$$y = y_1 \; y_2 \cdots y_n$$

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Bit Fixing Path BFP(x, y)

$$X = x_1\, x_2\, x_3 \cdots x_n$$
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$$y = y_1\, y_2 \cdots y_n$$

Algorithm

Step 1: For each $x \in V_n$ choose $\delta(x)$ independently and randomly from V_n — flip n coins for each x.

Step 2: Send packet p_x to $\delta(x)$ along the path $P(x) = BFP(x, \delta(x))$.

Step 3: Send packet p_x to $\pi(x)$ along the path $Q(x) = BFP[\delta(x), \pi(x)]$

Algorithm

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Step 3: Send packet P_x to $\pi(x)$ along the path $Q(x) = BFP[\delta(x), \pi(x)]$

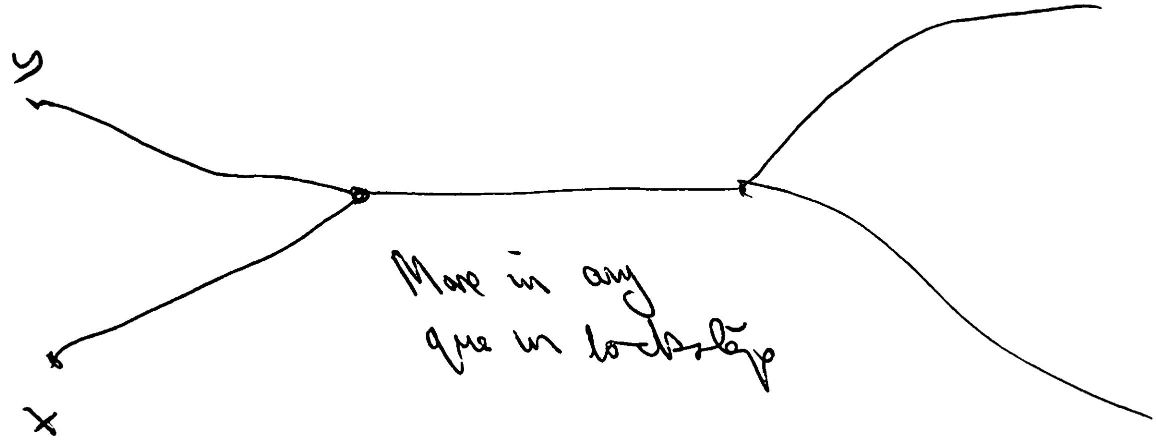
Let $D(x)$ be time spent by P_x waiting in a queue.

So P_x is delivered after at most $D(x)+n$ time steps.

$$S(x) = \{ y \neq x : P(x) \cap P(y) \neq \emptyset \}$$

<u>Claim</u>

$$S(x) \leq D(x)$$



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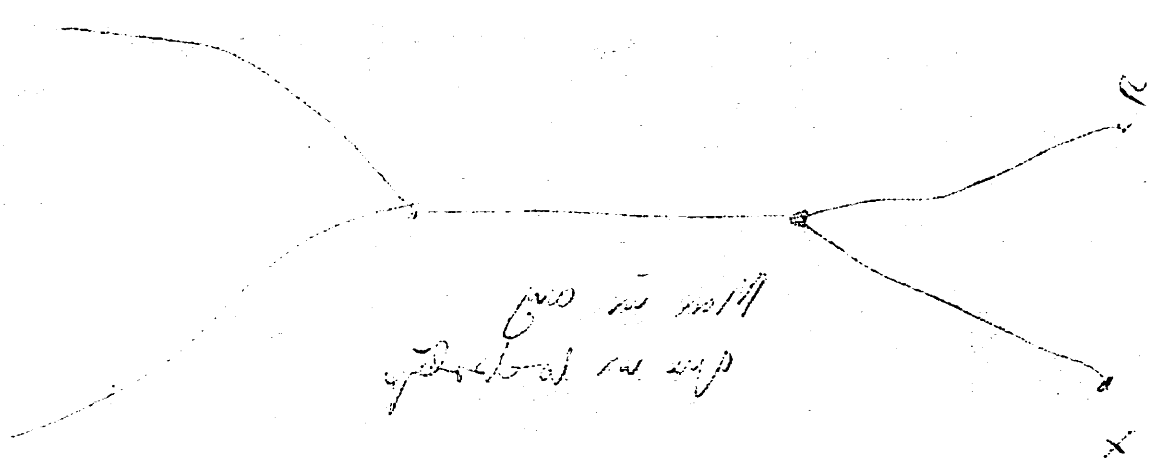

Claim:

$$P[\,|S(x)| \geq 3n\,] \leq 3^{-n} \quad (*)$$

$P[\exists x$: Step 2 takes more than $4n$ time$]$

$$\leq 2^n \times 3^{-n} \leq \left(\frac{2}{3}\right)^n.$$

$$|S(x)| = \sum_{y \neq x} Z_y$$

where $Z_y = \mathbb{1}_{P(x) \cap P(y) \neq \emptyset}$

So $|S(x)|$ is the sum of independent

0,1 random variable.

$E(|S(x)|) \leq \frac{1}{2} n$ — see slides

Apply Chernoff-Hoeffding to get $(*)$.

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