

9/18/15

Chernoff - Hoeffding Bounds

$$S_n = X_1 + X_2 + \dots + X_n \quad [X_i \text{ are independent}]$$

$$0 \leq X_i \leq 1$$

$i = 1, 2, \dots, n$

$$E(X_i) = \mu_i$$

We want to estimate

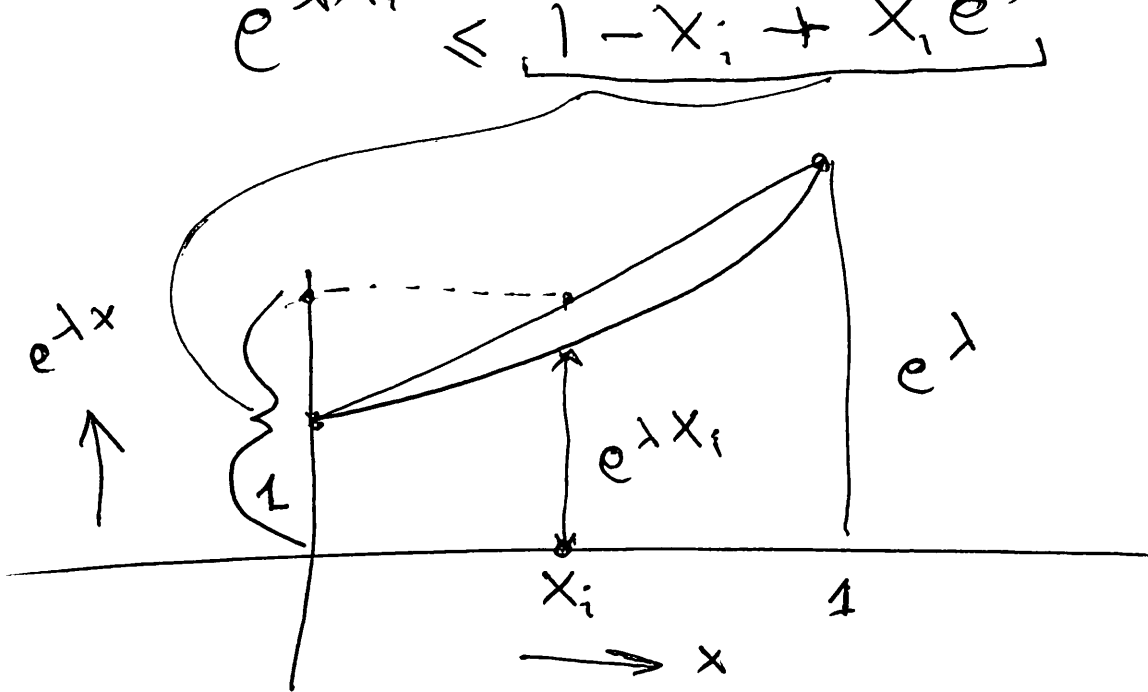
$$P[S_n \geq \mu + t]$$

where $\mu = \mu_1 + \mu_2 + \dots + \mu_n$.

$$\begin{aligned}
 P(S_n \geq \mu + t) &\leq e^{-\lambda(\mu+t)} E(e^{\lambda S_n}) \\
 &= e^{-\lambda(\mu+t)} \prod_{i=1}^n E(e^{\lambda X_i})
 \end{aligned}$$

Convexity implies that

$$e^{\lambda x_i} \leq \underbrace{1 - x_i + x_i e^{\lambda}}_{\text{chord}}$$



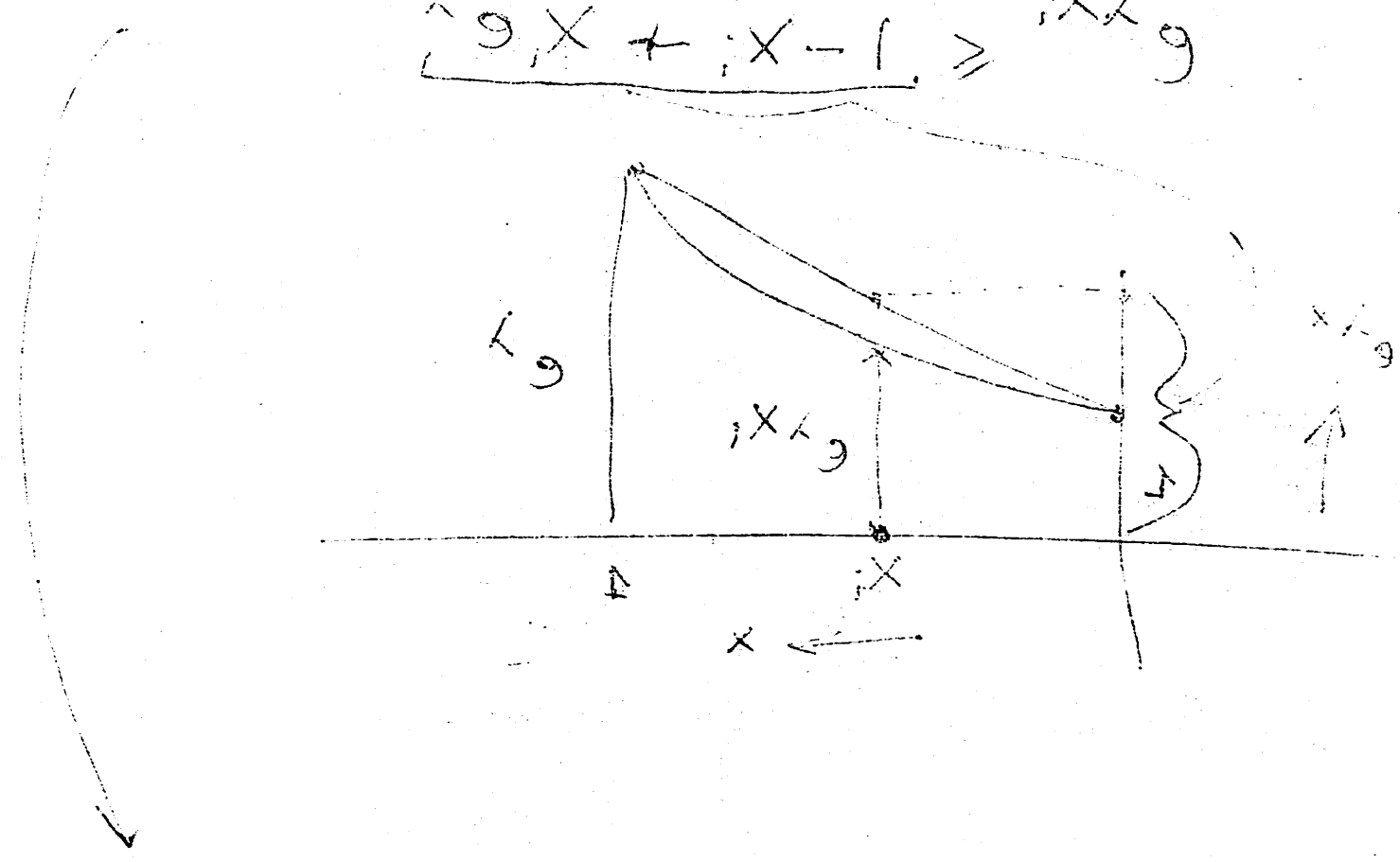
$$E(e^{\lambda X_i}) \leq 1 - \mu_i + \mu_i e^{\lambda}$$

$$p(x) = \sum_{k=0}^n \binom{n}{k} x^k = (1+x)^n$$

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Converting implies that

$$1 - x + x^2 \geq (1-x)^2$$



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$$P(\sum_{i=1}^n X_i \geq \mu + t) \leq e^{-\lambda(\mu+t)} \prod_{i=1}^n (1 - \mu_i + \mu_i e^\lambda)$$

$$\leq e^{-\lambda(\mu+t)} \left(\frac{n - \mu + \mu e^\lambda}{n} \right)^n$$

Geometric mean \leq arithmetic mean.

$$(x_1 x_2 \dots x_n)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \dots + x_n}{n}$$

Choose λ - calculus

$$e^\lambda = \frac{(\mu+t)(n-\mu)}{(n-\mu-t)\mu}$$

minimize

⋮

$$\left(\frac{(j+m+1) \prod_{i=1}^n (1-x_i)}{(j+m) \prod_{i=1}^n x_i} \right) \geq \left(\frac{j+m+1}{j+m} \right)$$

$$\left(\frac{(j+m+1) \prod_{i=1}^n (1-x_i)}{(j+m) \prod_{i=1}^n x_i} \right) \geq \left(\frac{j+m+1}{j+m} \right)$$

Geometric mean \geq arithmetic mean

$$\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n}$$

(Proof by induction)

$$\frac{(n-1)(j+m)}{n(j+m-1)} = \frac{j+m}{j+m-1}$$

induction

$$P(S_n \geq \mu + t) \leq \exp\left\{-\frac{t^2}{2(\mu + \frac{t}{3})}\right\}$$

$$P(S_n \leq \mu - t) \leq \exp\left\{-\frac{t^2}{2(\mu - \frac{t}{3})}\right\}$$

Remember these: $t = \epsilon \mu$

$$P(S_n \geq (1 + \epsilon)\mu) \leq e^{-\frac{\mu \epsilon^2}{3}}$$

$$P(S_n \leq (1 - \epsilon)\mu) \leq e^{-\frac{\mu \epsilon^2}{2}}$$

$$P(S_n \geq c\mu) \leq \left(\frac{c}{2}\right)^{c\mu}$$

$$\left\{ \frac{s_j}{(j+1)C} \right\} \text{exp} \geq (j+1) \leq 2 \quad \text{q}$$

$$\left\{ \frac{s_j}{(j-1)C} \right\} \text{exp} \geq (j-1) \geq 2 \quad \text{q}$$

$j = \epsilon$: well remember

$$\frac{s_{\epsilon}}{\epsilon} \text{exp} \geq (\epsilon+1) \leq 2 \quad \text{q}$$

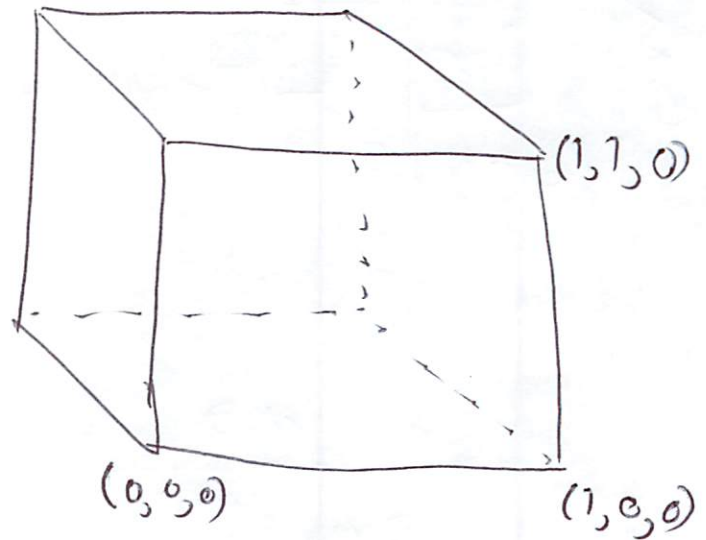
$$\frac{s_{\epsilon}}{\epsilon} \text{exp} \geq (\epsilon-1) \geq 2 \quad \text{q}$$

$$\frac{s_{\epsilon}}{\epsilon} \text{exp} \geq (\epsilon) \leq 2 \quad \text{q}$$

Valiant-Brebner Routing Algorithm

$Q_n = n$ -cube

$$V_n = \{0,1\}^n$$

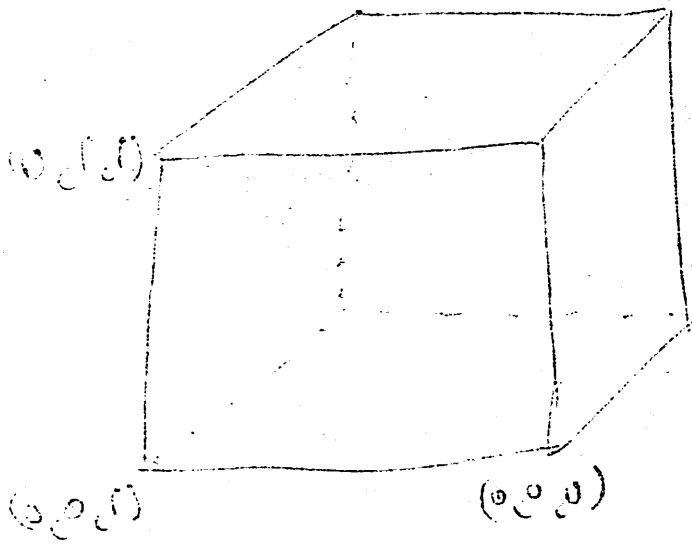


Each $x \in V_n$ wants to send a packet from $x \rightarrow \pi(x)$ where π is a permutation.

Problem is that one one packet can cross an edge at one time.

Send the packets synchronously - each time step on any ≤ 1 packet crosses

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$$Q_{cl} = \int \dots$$

$$\sum_{i=1}^N \dots = N V$$

Energy $X \in V$ wants to be in a box

$$X \rightarrow \pi(X) \text{ where } \pi$$

permutation

Problem: What are the boxes we

cross on edge of our phase

space - phase space all boxes

are equal and we get the same

We need a randomised algorithm -
it is known that for any deterministic
routing algorithm, there will be
bad permutations that take a long
time to deliver.

Bit Fixing Path

BFP(x, y)

$$X = x_1 \ x_2 \ x_3 \ \dots \ x_n$$

↓

$$y_1 \ x_2 \ x_3 \ \dots \ x_n$$

↓

$$y_1 \ y_2 \ x_3 \ \dots \ x_n$$

⋮

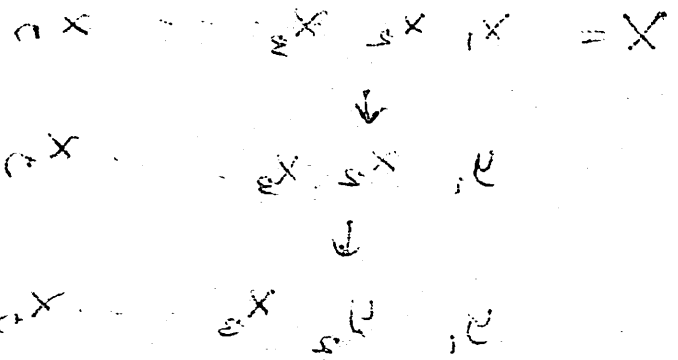
⋮

$$y = y_1 \ y_2 \ \dots \ y_n$$

- multiple summations a few are
 known for only algorithms
 and how many multiple points
 and a lot of calculations had
 to be done.

BFB (X, N)

BFB Fixed Path



$$x^2 + x^3 + x^4 + x^5 = x^2$$

Algorithm

Step 1: For each $x \in V_n$ choose $S(x)$ independently and randomly from V_n — flip n coins for each x .

Step 2: Send packet p_x to $S(x)$ along the path $P(x) = \text{BFF}(x, S(x))$.

Step 3: Send packet p_x to $\pi(x)$ along the path $Q(x) = \text{BFF}[S(x), \pi(x)]$

Algebra

Step 1: For each $x \in V$, choose $\phi(x)$ independent and random
from V — $\phi(x)$
for each x .

Step 2: For each $x \in V$, choose $\phi(x)$ also the pair $(x, \phi(x))$

Step 3: For each $x \in V$, choose $\phi(x)$ the pair $(x, \phi(x))$

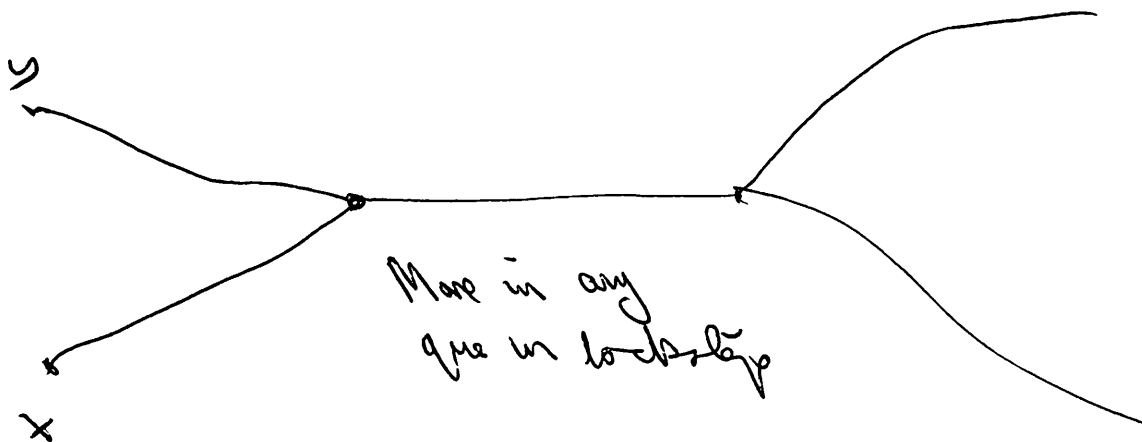
Let $D(x)$ be time spent by P_x waiting in a queue.

So P_x is delivered after at most $D(x) + n$ time steps,

$$S(x) = \{y \neq x : P(x) \cap P(y) \neq \emptyset\}$$

Claim

$$S(x) \leq D(x)$$



Claim:

$$P[|S(x)| \geq 3n] \leq 3^{-n} \quad (*)$$

$P[\exists x: \text{Step 2 takes more than } 4n \text{ time}]$

$$\leq 2^n \times 3^{-n} \ll \left(\frac{2}{3}\right)^n.$$

$$|S(x)| = \sum_{y \neq x} Z_y$$

where $Z_y = \frac{1}{P(x) \cap P(y)} \neq \emptyset$

So $|S(x)|$ is the sum of independent
0,1 random variables.

$$E(|S(x)|) \leq \frac{1}{2} n \quad - \text{ see slides}$$

Apply Chernoff-Hoeffding to get $(*)$.

(1) $P(|Z(x)| \geq \epsilon) \leq 3^{-n}$

[and estimate error from set $\{x \in \mathcal{X} : |Z(x)| \geq \epsilon\}$]

$$\sum_{x \in \mathcal{X}} \mathbb{1}_{\{|Z(x)| \geq \epsilon\}} \leq \sum_{x \in \mathcal{X}} 3^{-n}$$

$$\sum_{x \neq c} |Z(x)| = \sum_{x \in \mathcal{X}} |Z(x)| - |Z(c)|$$

$$\sum_{x \neq c} |Z(x)| \leq \sum_{x \in \mathcal{X}} |Z(x)| = \sum_{x \in \mathcal{X}} \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{x \neq c\}} = \frac{1}{n} \sum_{i=1}^n (n-1) = n-1$$

Induction hypothesis: $\sum_{x \neq c} |Z(x)| \leq \frac{1}{2} n$

Induction hypothesis: $\sum_{x \neq c} |Z(x)| \leq \frac{1}{2} n$

$$E(|Z(x)|) \leq \frac{1}{2} n$$

(2) $E(|Z(x)|) \leq \frac{1}{2} n$