9\14\15

Intersection Safe Families

$\mathcal{A} \subseteq 2^{[n]}$ = power set of $[n]$

is intersection safe, if $\forall$ distinct $A, B, C \in \mathcal{A}$

$\Rightarrow A \cap B \nsubseteq C$.

How can we choose a large "safe" family?

We choose $X_1, X_2, \dots, X_p$ randomly.

Z = # of triples i, j, k such that

$X_i \cap X_j \subseteq X_k$.

We choose p so that $E(Z) < 1$

$\Rightarrow \exists \mathcal{A}$ of size p such that $\mathcal{A}$ is safe.

If all families of size p are unsafe

then $Z \geq 1$, $\forall$ families of size p

and then $E(Z) \geq 1$.

How does one choose a random subset of $[n]$?

We <u>independently</u> flip a coin for each $i \in [n]$.

$$\Pr[(X_i \cap X_j) \subseteq X_k] = \left(\frac{7}{8}\right)^n$$

$$E(Z) = p(p-1)(p-2)\left(\frac{7}{8}\right)^n.$$

So if $p \leq \left(\frac{8}{7}\right)^{n/3}$ then $EZ < 1$

and a family exists.

Minor Problem: might choose $X_i = X_j$ and not get number of sets claimed.

$Z_1 = \#\, i,j : X_i = X_j$

Need to get $E(Z + Z_1) < 1$.

$E(Z_1) = \binom{p}{2} \frac{1}{2^n}$

We need

$$p(p-1)(p-2)\left(\frac{7}{8}\right)^n + \frac{p(p-1)}{2} \frac{1}{2^n} < 1$$

$p \approx \left(\frac{8}{7}\right)^{\frac{n}{3}}$ still.

Inequalities

Markov Inequality: $X : \Omega \to \{0, 1, 2, \dots\}$

Assume $\mu = E(X) < \infty$

For any $t \geq 1$

$$\Pr[X \geq t] \leq \frac{E(X)}{t}$$

Proof

$$E(X) = E(X \mid X \geq t) \Pr(X \geq t) + E(X \mid X < t) \Pr[X < t]$$
$$\geq t \Pr[X \geq t]$$

$\square$

In particular $\Pr[X \neq 0] = \Pr[X \geq 1] \leq E(X)$.
(First moment method)

Chebyshev Inequality

Suppose $\sigma = \sqrt{Var(Z)} < \infty$

$$\Pr[\,|Z-\mu| \geq t\,\sigma\,]$$

$$= \Pr[\,(Z-\mu)^2 \geq t^2\sigma^2\,]$$

$$\leq \frac{E((Z-\mu)^2)}{t^2\sigma^2} \quad \text{Markov}$$

$$= \frac{1}{t^2}.$$

Binomial Distribution

$B = B(n,p)$: $\quad \Pr[B = k] = \binom{n}{k} p^k (1-p)^{n-k}$

$$E(B) = np$$

Var(B):

$$E(B^2) = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{\substack{\cancel{k=0} \\ k=2}}^{n} k(k-1) \binom{n}{k} p^k (1-p)^{n-k} + \sum_{\substack{\cancel{k=0} \\ k=1}}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$n(n-1) \sum_{k=2}^{n} \binom{n-2}{k-2} p^k (1-p)^{n-k} + n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= n(n-1)p^2 \sum_{l=0}^{n-2} \binom{n-2}{l} p^l (1-p)^{n-2-l} + np \sum_{l=0}^{n-1} \binom{n-1}{l} p^l (1-p)^{n-1-l}$$

$l = k-2 \qquad\qquad l = k-1$

$$= n(n-1)p^2 - np$$

$$E(B^2) = n(n-1)p^2 + np$$

$$Var(B) = E(B^2) - E(B)^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= np(1-p).$$

Apply Chebyshev

$$Pr[|B_{n,p} - np| \geq \epsilon np] \leq \frac{1}{\epsilon^2 np}$$

[Law of large numbers]