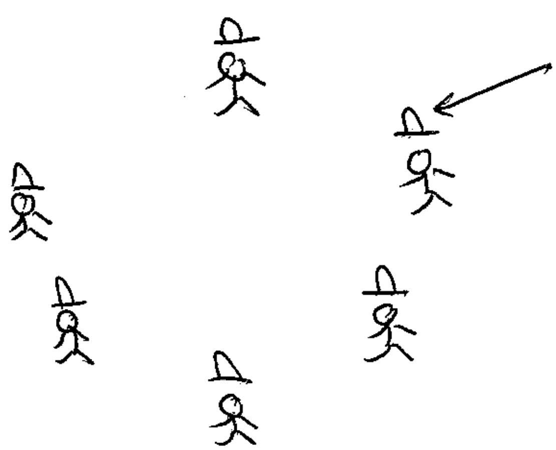
9/11/15

A problem with hats.

n people in a circle.



Each person can see color of everybody's hat except their own.

Each person asked to give color of their hat.

(i) Pass

(ii) Get it right — all get it right - big prize

(iii) Get it wrong — one person gets it wrong - disaster

Partition $Q_n = \{0,1\}^n$ into two sets W, L with the following property:

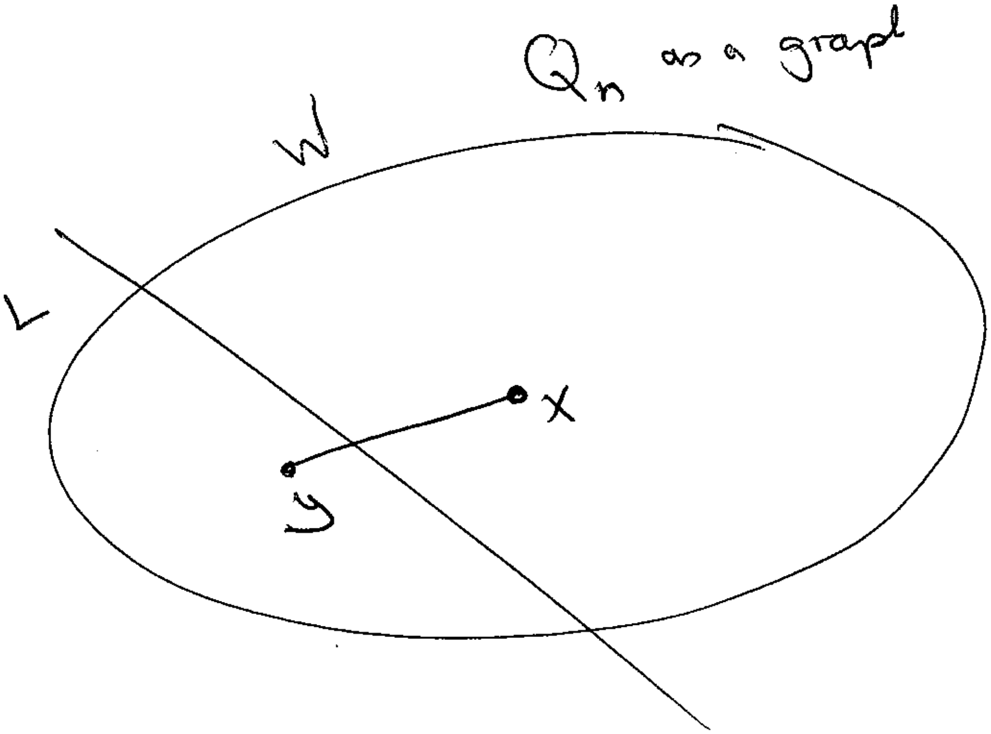
$$x \in W \Rightarrow \exists\, y \in L$$

$x = x_1 x_2 \cdots x_n$ $\qquad$ $y = y_1 y_2 \cdots y_n$

such that $h(x,y) = 1$

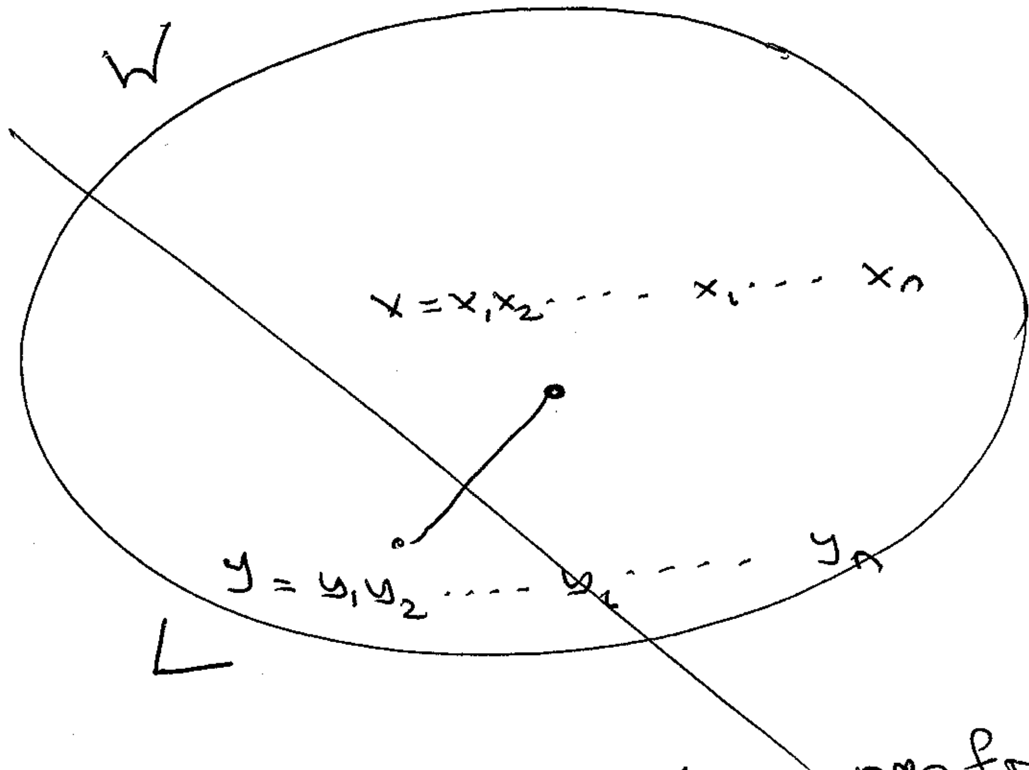
$$h(x,y) = |\{j : x_j \neq y_j\}|$$

= Hamming distance between x & y.



The people agree on a set W before they put on their hats

Assume $x \in W$ $\left[\begin{array}{l}\text{Blue} = 1 \\ \text{Red} = 0\end{array}\right]$



$\exists\, i$: switching x_i to y_i goes from x to y.

So i choose color that places x in W.

L is a cover - <u>dominating set</u>

There is a small cover.

Let $p = \frac{\ln n}{n}$. Choose L_1 randomly by placing $y \in Q_n$ into L_1 with probability p.

L_2 is those $z \in Q_n$ not adjacent $(h=1)$ to anything in L_1.

Take $L = L_1 \cup L_2$

$$E(|L|) = 2^n p + 2^n (1-p)^{n+1}$$

$(1_{x \in L_2}$ if x & all nbrs of x are not in $L_1)$

$$\leq 2^n p + 2^n e^{-np} \qquad [1-x \leq e^{-x}$$

$$\leq 2^n \times \frac{2 \log n}{n} \qquad e^{-np} = \frac{1}{n}$$

Because $E(|L|) \leq 2^{n+1} \frac{\log n}{n}$ we know that there exists L with $|L| \leq 2^{n+1} \frac{\log n}{n}$.

First Moment Method

Let X be a random variable that takes values in $\{0, 1, 2, \ldots\}$.

Then

$$\Pr(X \geq 1) \leq E(X)$$

<u>Proof.</u>

$$E(X) = \overbrace{E(X \mid X = 0)}^{0} \Pr[X = 0] + \underbrace{E(X \mid \geq 1)}_{\geq 1} \Pr(X \geq 1)$$

$$\geq \Pr(X \geq 1)$$

Intersection Safe Families

$\mathcal{A}$ is a family of sub-sets of $[n]$ such that for distinct $A, B, C \in \mathcal{A}$,
$C \not\supseteq A \cap B$.

Using the probabilistic method we find an intersection safe family of <u>exponential size</u>