12\4\2015

D = domain

C = {colors}

$X = \{x : D \to C\}$

$= \{\text{colorings of } D\}$

G is now a set of permutation of D.

g permutes D.

We need to figure out how g permutes X

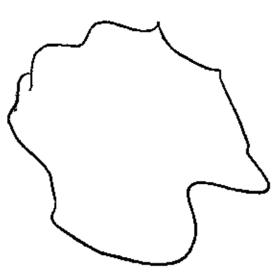
We need to define the coloring $g * x$

We need to define $g * x(d)$ for all $d \in D$

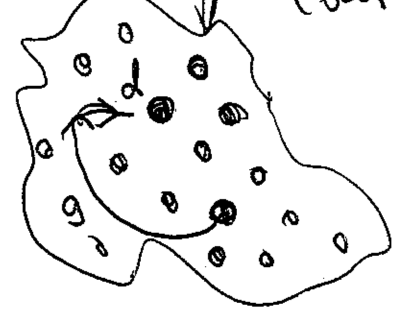
$$g * x(d) = x(g^{-1}(d))$$

each • has a color under x

D =



g rotates D



$$\begin{array}{|c|c|} \hline R & B \\ \hline R & B \\ \hline \end{array} \equiv R^2 B^2 \equiv \begin{array}{|c|c|} \hline R & B \\ \hline B & R \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline R & B \\ \hline B & B \\ \hline \end{array} \equiv R B^3$$

We associate a <u>weight</u> w_c with each $c \in C$.

If $x \in X$ then

$$W(x) = \prod_{d \in D} w_{x(D)}$$

$S \subseteq X$ we let <u>inventory</u> of S

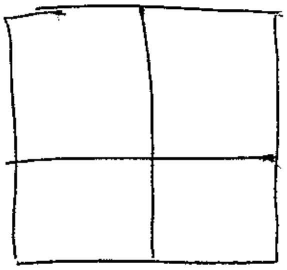
$$W(S) = \sum_{x \in S} W(x)$$

Pattern Inventory

$$PI = W(S^*)$$

S^* contains one coloring from each orbit.

2×2 Chessboard.



$$PI = R^4 + R^3B + 2R^2B^2 + RB^3 + B^4$$

<u>Lemma</u>

If x & y are in the same orbit then $W(x) = W(y)$

<u>Proof</u>

Suppose $g*x = y$.

$$W(y) = \prod_{d \in D} W_{y(d)}$$

$$= \prod_{d \in D} W_{g*x(\cdot d)}$$

$$= \prod_{d \in D} W_{x(g^{-1}(d))}$$

$$= \prod_{d \in D} W_{x(d)}$$

$$= W(x).$$

Let $\Delta = |D|$. If $g \in G$ has k_i cycles of length i then

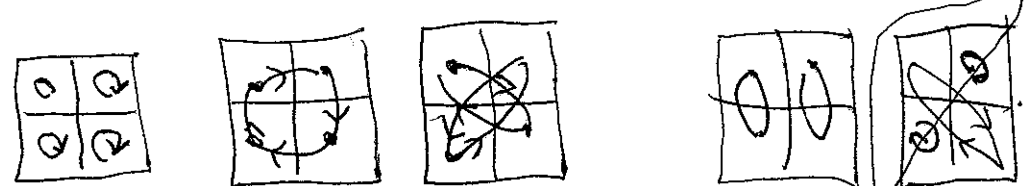
$$ct(g) = x_1^{k_1} x_2^{k_2} \cdots x_\Delta^{k_\Delta}$$

Cycle Index Polynomial of G, C_G

$$C_G = \frac{1}{|G|} \sum_{g \in G} ct(g)$$

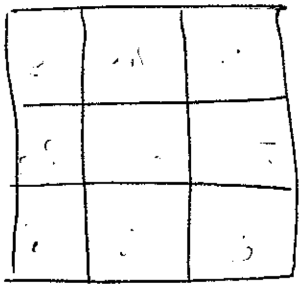
<u>Example 2</u>

g	e	a	b	c	p	q	r	s
$ct(g)$	x_1^4	x_4	x_2^2	x_4	x_2^2	x_2^2	$x_1^2 x_2$	$x_1^2 x_2$



r

$$C_G(x_1, x_2, x_3, x_4) = \frac{1}{8}\left(x_1^4 + 3x_2^2 + \cdots \right)$$



$$\begin{array}{cccccccc} e & a & b & c & p & q & r & s \\ x_1^9 & x_1x_4^2 & x_1x_2^4 & x_1x_4^2 & x_1^3x_2^3 & x_1^3x_2^3 & x_1^3x_2^3 & x_1^3x_2^3 \end{array}$$

$$C_G = \frac{1}{8}\left(x_1^9 + x_1x_2^4 + 4x_1^3x_2^3 + 2x_1x_4^2\right)$$

Polya

$$PI = C_G\left(\sum_{c\in C} w_c, \sum_{c\in C} w_c^2, \dots \right)$$

Suppose $C = \{R, B, G\}$

$$C_G = \frac{1}{8}\left((R+B+G)^9 + (R+B+G)(R^2+B^2+G^2)^4 + \dots\right.$$

$X = X_1 \cup X_2 \cup \dots X_m$ equivalence classes

$X_i = \bigcup$ orbits

$x \sim y$ iff $W(x) = W(y)$.

size of class

$$PI = \sum_{i=1}^{m} m_i W_i$$

weight of a member of class

$$= \sum_{i=1}^{m} \left(\frac{1}{|G|} \sum_{g \in G} |Fix(g^{(i)})| \right)$$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{m} |Fix(g^{(i)})| \, W_i$$

$$Fix(g) = \bigcup_{i=1}^{m} Fix(g^{(i)})$$

$$= \frac{1}{|G|} \sum_{g \in G} W(Fix(g))$$

Polya formula

$W($ Fix()

$$(R^4 + B^4 + G^4) \times (R^5 + B^5 + G^5) \times (R^2 + B^2 + G^2) \times (R + B + G)$$