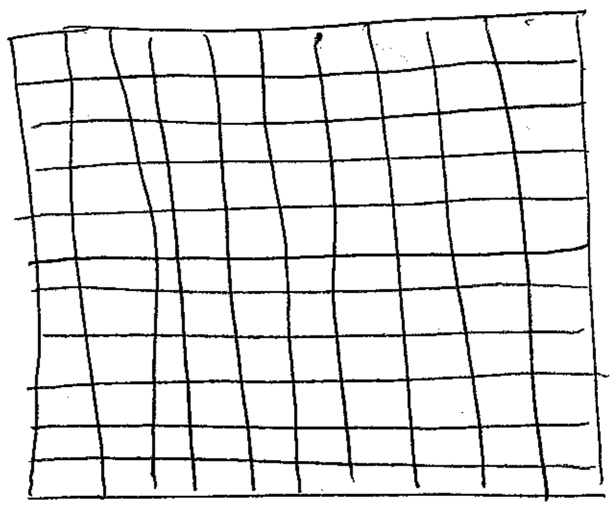
12/2/15

$$\underbrace{\nu_{X,G}}_{\#\text{ of orbits}} = \frac{1}{\underbrace{|G|}_{\text{group}}} \sum_{x \in \underbrace{X}_{\text{colorings}}} |S_x| \qquad (*)$$



$n \times n$

chessboard

2 colors

$$G = \{e, \underbrace{a, b, c}_{\text{rotations}}, \underbrace{p, q}_{\text{Reflection in } - \text{ or } |}, \underbrace{r, s}_{\text{Reflection in } \backslash \text{ or } /}\}$$

$|X| = 2^{n^2} \Rightarrow (*)$ is difficult to apply

Suppose $g \in G$.

$$\mathrm{Fix}(g) = \{ x \in X : g * x = x \}$$

$$= \{ x : x \text{ is fixed by } g \}.$$

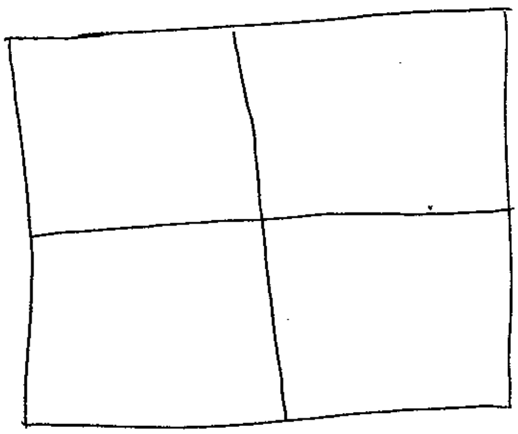
<u>Frobenius/Burnside</u>

$$\nu_{X,G} = \frac{1}{|G|} \sum_{g \in G} | \mathrm{Fix}(g) |$$

<u>Proof</u> Claim $\displaystyle\sum_{x \in X} |S_x| = \sum_{g \in G} |\mathrm{Fix}(g)|$

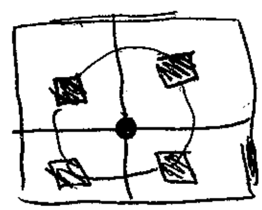
$$\begin{array}{cc} & \rightarrow X \qquad\qquad x \\ \downarrow G & \\ g. & \left[\begin{array}{ccc} & & \\ & & \\ & \mathbb{1}_{g*x=x} & \\ & & \\ & & \end{array} \right] \#1\text{'s} = |\mathrm{Fix}(g)| \end{array}$$

$$\#1\text{'s} = |S_x|$$

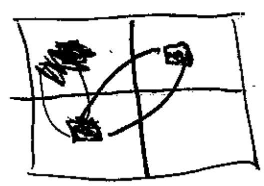


$n \times n$ chessboard
n even

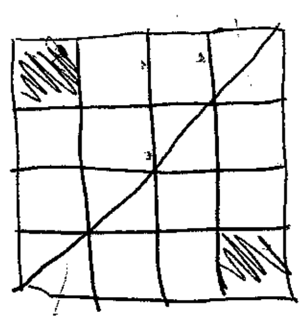
g	$\lvert fix(g) \rvert$
e	2^{n^2}
$a = 90°$	$2^{n^2/4}$
$b = 180°$	$2^{n^2/2}$
$c = 270$	$2^{n^2/4}$
p	$2^{n^2/2}$
q	$2^{n^2/2}$
r	$2^{n(n+1)/2}$
s	$2^{n(n+1)/2}$



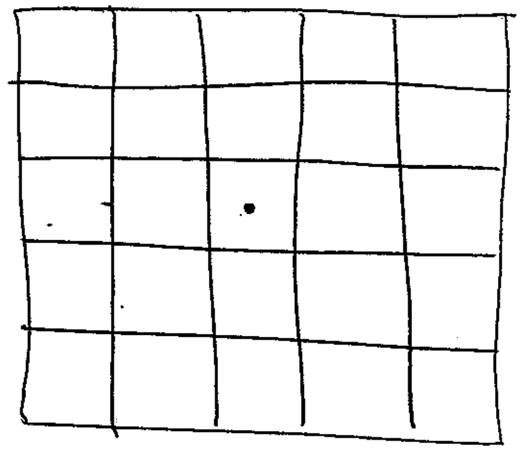
← a



b



Odd $n=5$



n odd
slightly different
but just as
easy to do.

If there are q colors then we just replace
2 by q.

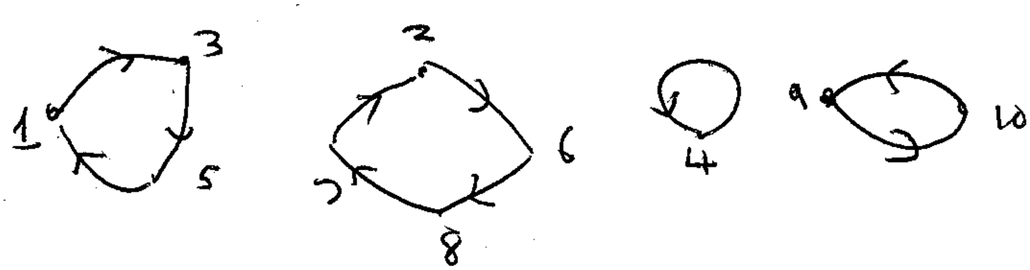
$$|Fix(g)| = q^{\#\text{ of cycles of } g}$$

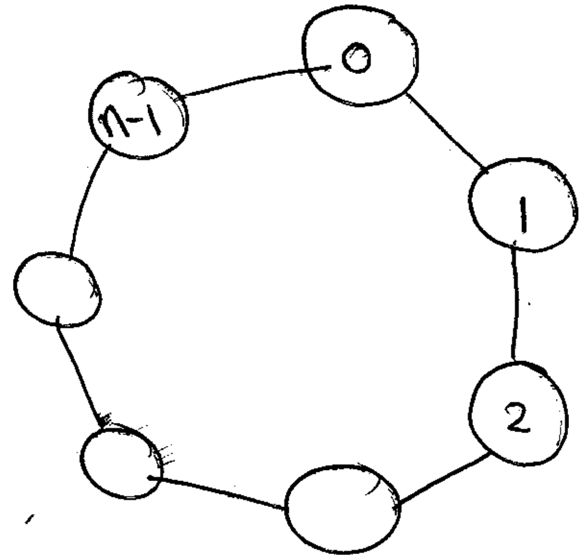
g is a permutation of X.

Given a permutation π on X we have a
digraph $D_\pi = (X, \{(x, \pi(x)) : x \in X\})$

$X = [10]$

x	1	2	3	4	5	6	7	8	9	10
$\pi(x)$	3	6	5	4	1	8	2	7	10	9





G = group of rotations $e_0, e_1, \dots, e_{n-1}$

e_j rotates by $\frac{2\pi j}{n}$

e_j move $k \to k+j$

q colors

$$\nu_{X,G} = \frac{1}{n} \sum_{m=0}^{n-1} q^{|\mathrm{Fix}(e_m)|}$$

$$= \frac{1}{n} \sum_{m=0}^{n-1} q^{\#\text{ of cycles in } e_m}$$

Fix m and consider the cycle C_i containing i

$$C_i = \{ i, i+m, i+2m, \dots, i+(k_m-1)m \}$$

where k_m is the smallest integer such that $m k_m$ is divisible by n.

$$k_m = \frac{n}{a_m} \quad \text{where } a_m = \gcd(m,n)$$

All you need is $C_0, C_1, \ldots, C_{a_m - 1}$

$$\nu_{X,G} = \sum_{m=0}^{n-1} q^{\gcd(m,n)}$$

Suppose we have colors Red, Blue, Green.

Burnside / Frobenius tells us how many distinct colorings there are.

Polya refines this and tells us how many distinct colorings there are with n_R Red, n_B Blue and n_G Green.