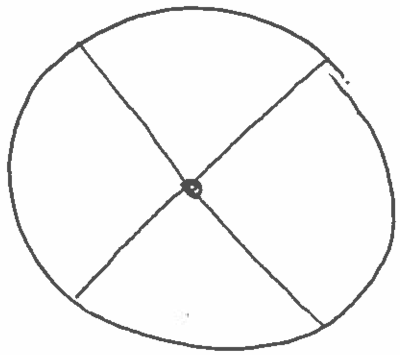
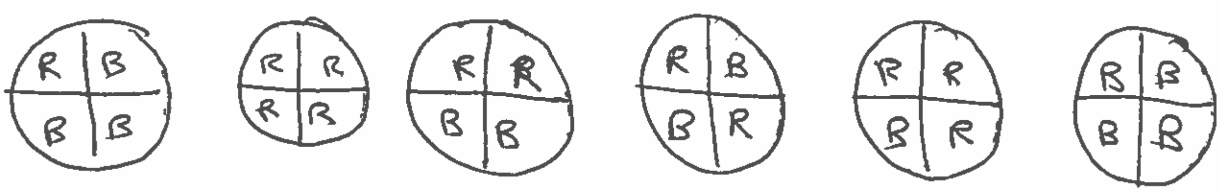
11/30/15



Disk can spin around center

How many distinct ways are there of coloring the sectors Red & Blue.

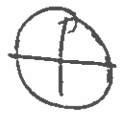
Two colorings are "the same" if you can from one to the other by a rotation.




Only 6 colorings

# General situation

We have a set $X$ which will stand for the set of colorings, when transformations are not allowed

$|X| = 2^4$ in  .

In addition there is a set $G$ of permutations of $X$. This set $G$ will have a <u>group</u> structure

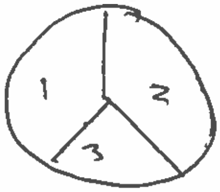
Given two members $g_1, g_2 \in G$ we define $g_1 \circ g_2$ where $g_1 \circ g_2 (x) = g_1(g_2(x))$

We insist that $g_1 \circ g_2 \in G$.

A1: Identity $1_X \in G$ $\qquad 1_X(x) = x$

A2: $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$

A3: $g^{-1} \in G$ for all $g \in G$ where $g^{-1} \circ g = 1_X$




$X = \{ RRR,\ RRB,\ RBR,\ RBB,\ BRR,\ BRB,\ BBR,\ BBB \}$

$G = e_0,\ e_1\ \ e_2$

identity

rotate by $120^\circ = \frac{2\pi}{3}$

rotate by $240^\circ$ or $\frac{4\pi}{3}$

We use notation $g * x$ instead of $g(x)$

We are talking about a <u>group action</u>

We say that two colorings $x$ & $y$ are "the same" if there exists $g \in G$ such that $g * x = y$

$e_1 * RRB = BRR$ so $RRB$ & $BRR$ are the same.

If $x \in X$ then its <u>orbit</u> $O_x$ is

$$O_x = \{y \in X : \exists g \in G \text{ such that } g * x = y\}$$

<u>Lemma 1</u>

The orbits partition $X$.

<u>Proof</u>

(i) $x = 1_X * x$ and so $x \in O_x$

(ii) Suppose $O_x \cap O_y \neq \emptyset$ ie $\exists\, g_1, g_2$ such that

$$g_1 * x = g_2 * y$$

Suppose $g \in G$

$$g * x = g \circ g_1^{-1} \circ g_2 * y$$

$\Longrightarrow$ $O_x \subseteq O_y$ & similarly $O_y \subseteq O_x$

The stabilizer $S_x = \{ g : g * x = x \}$

[It is a subgroup of $G$]

Lemma 2

If $x \in X$ then $|O_x| \, |S_x| = |G|$

Proof

Fix $x \in X$ and define an equivalence relation $\sim$ on $G$ by

$$g_1 \sim g_2 \quad \text{iff} \quad g_1 * x = g_2 * x$$

Let the equivalence classes be $A_1, A_2, \dots, A_m$.

(i) $m = |O_x|$

(ii) We show that $|A_i| = |S_x|$.

Fix $i$ and $g \in A_i$

$$\begin{aligned} h \in A_i &\leftrightarrow g * x = h * x \\ &\leftrightarrow g^{-1} \circ h \in S_x \\ &\leftrightarrow h \in g \circ S_x \end{aligned}$$

$|A_i| = |g \circ S_x| = |S_x|$.

Suppose $g \circ \sigma_1 = g \circ \sigma_2$
$\Rightarrow \sigma_1 = \sigma_2$

$\nu_{X,G} = \#$ orbits

## Thm 1

$$\nu_{X,G} = \frac{1}{|G|} \sum_{x \in X} |S_x|$$

## Proof

$\{1,2\}\ \{3,4,5\}\ \{6\}$

$3 = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + 1$

$$\nu_{X,G} = \sum_{x \in X} \frac{1}{|O_x|}$$

$$= \sum_{x \in X} \frac{|S_x|}{|G|}.$$

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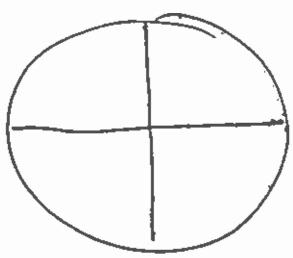


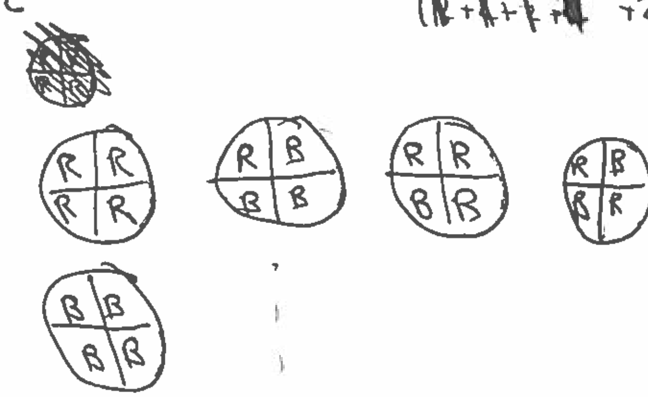
$|X| = 16$

$|G| = 4$

Rotations

$\frac{1}{4}(8 + 8 + 8) = 6$

$\frac{1}{4}\left[(4+4) + (1+1+1+1+1+1+1+1) + (+2+2)\right]$




2 8 4 2