11\23\15

$A_1, A_2, \dots, A_N$ are the sets.

We proved that if $|A_i| \geq n$ & each $x \in A$ is in $\leq \frac{n}{2}$ of the A_i then Player 2 can draw via a pairing strategy

Apply this to $\overbrace{n \times n \times \cdots \times n}^{d}$ Tic Tac Toe.

(I) $|A_i| = n$ for all i.

(II) # of lines through a fixed point $(c_1, c_2, \dots, c_d)$

n odd: $\frac{3^d - 1}{2}$

Only happens if $c_i = \frac{n+1}{2}, \forall i$

$\dots \; c_i \; \dots$

line is either constant / decreasing / increasing on i

n even: once we fix places where component is constant, line is determined

$\underline{d=4}$

$(\; x \quad y \;)$

fixed   fixed

$(\dots \; 4 \; \dots \; 6 \; \dots \; 4 \;) \; \dots$

5	5
6	4
7	3

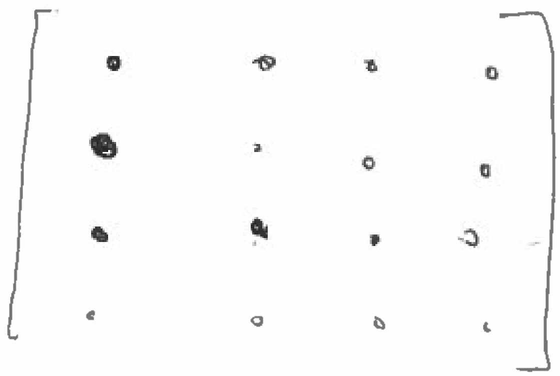
$\underline{n \text{ is even}}$ $\quad x \leq n/2$

$\underline{n=10}$ $\quad x=4 \quad y=6$

n even: $\# \text{lines} \leq \sum_{i=0}^{d-1} \binom{d}{i} - 1 = 2^d - 1$

places where not constant

4×4



$c_1 \quad c_2 \quad \cdots \quad c_d$

$3 \quad 7 \quad \cdots$ fixed

$$\frac{n}{2} \geq \frac{3^d - 1}{2} \quad \text{or} \quad \frac{n}{2} \geq 2^d - 1$$

n odd   n even

3	1
3	.
3	.
3	7
.	8
.	:
3	n

1	7	- - -
2	7	- - -
3	7	- - -
4	7	- - -
⋮		
n	7	- - -

Odd

$f \quad f \quad \frac{n+1}{2} \quad f \quad f \quad \frac{n+1}{2} \quad f \quad f \quad f$

decreas   increase

3^d

Even

$$
\begin{array}{ccccccccc}
f & f & x & f & f & y & \cdots & z \\
 & & x-1 & & & y-1 & & z+1 \\
 & & x-2 & & & \vdots & & \vdots \\
 & & \vdots & & & \vdots & & \vdots \\
 & & \underline{1} & & & 1 & & n
\end{array}
$$

$x \leq \frac{n}{2}$

$\Rightarrow y = x$

$$\Longrightarrow n - z + 1 = x$$

$$z = n - x + 1$$

$$> x$$

~~If~~ So if d is fixed and we make n large enough, then player 2 can draw.

If n is fixed and we make d large enough then player 1 will win.

Hales-Jewitt Theorem.

~~For~~ If we r-color $[n]^d$ and d is sufficiently large then there is a mono-chromatic line. — Ramsey statement.

Tic Tac Toe — $r=2$

Suppose we go on until every point is colored. If d is large then there must be a line of one color.

Erdös-Selfridge Theorem

If $|A_i| \geq n$ for $i \in [N]$ and $N < 2^{n-1}$ then Player 1 can get a draw.

Proof

At any point in the game let C_j denote the set of elements of A which have been colored $j = 1, 2$. $U = A \setminus (C_1 \cup C_2)$.

$$\Phi = \sum_{i : A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|}$$

(= expected number of "bad sets for 2" if 2 randomly colored.)

Suppose that the players choices are $x_1, y_1, x_2, y_2, \ldots$

After first move $\Phi \leq N 2^{-(n-1)} < 1$

Now it's player 2's turn.

Claim: Player 2 can keep $\Phi < 1$ for the whole game.

$$\Downarrow$$

at the end $\displaystyle\sum_{i:\, A_i \cap C_2 = \emptyset} 1 < 1$

$\Rightarrow \nexists\, i : A_i \cap C_2 = \emptyset$.

Φ_j = value of Φ after $x_1 y_1 x_2 y_2 \cdots x_j$

move = $y_j x_{j+1}$

$$\Phi_{j+1} - \Phi_j = -\sum_{\substack{i:\, A_i \cap C_2 = \emptyset \\ y_j \in A_i}} 2^{-|A_i \cap U|} + \sum_{\substack{i:\, A_i \cap C_2 = \emptyset \\ y_j \notin A_i,\, x_{j+1} \in A_i}} 2^{-|A_i \cap U|}$$

$$\le -\sum_{\substack{i \in A_i \cap C_2 \\ y_j \in A_i}} 2^{-|A_i \cap U|} + \sum_{\substack{i:\, A_i \cap C_2 = \emptyset \\ x_{j+1} \in A_i}} 2^{-|A_i \cap U|}$$

≤ 0 if Player 2 chooses y_j to maximise