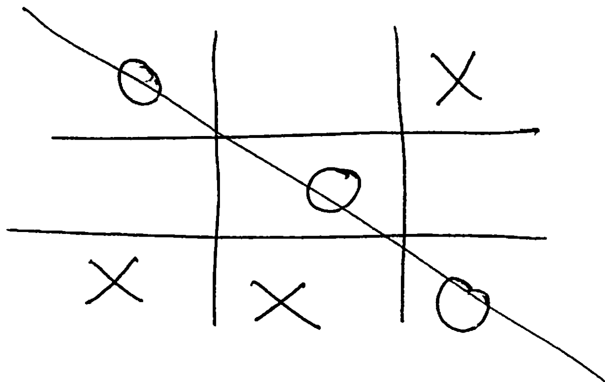
11/20/2015

Tic Tac Toe & Generalisations



The **board** consists of $[n]^d$

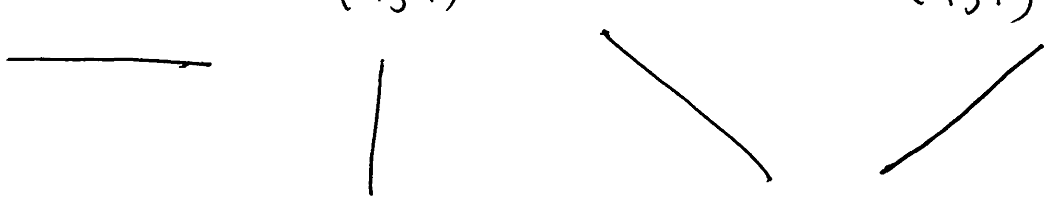
A **line** is a set of points

$$(x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(d)}), \quad j = 1, 2, \ldots, n$$

where each sequence of components $x^{(i)}$ is either (i) $k, k, k, \ldots, k$ for some $k \in [n]$ or (ii) $1, 2, \ldots, n$ or (iii) $n, n-1, n-2, \ldots, 1$

$d=2$
$n=4$

(1,1)		(1,1)		(1,1)		(1,4)
(1,2)	or	(2,1)	or	(2,2)		(2,3)
(1,3)		(3,1)		(3,3)		(3,2)
(1,4)		(4,1)		(4,4)		(4,1)



Lemma 1

The number of winning line is

$$\frac{(n+2)^d - n^d}{2}$$

$$X_j^{(1)}, X_j^{(2)}, \ldots, X_j^{(d)} \qquad j = 1, 2, \ldots, n$$

$\uparrow$

n choices of (1)

$+$

1 (11)

$+$

1 (111)

$-n^d$ ~~1, 1, 1~~,

1	2	4	10
1	2	4	10
1	2	4	10
1	2	4	10

Same point repeated n times.

Two players. Red player (X player) and Blue player, each take turns occupying (coloring) a point. A player wins if he/she colors a complete line.

__Draw__ if nobody colors a line.

Lemma:

Player 1 can always get a draw or better.

Proof

Strategy Stealing: If player 2 has a winning strategy, then player 1 could make an arbitrary first move and then win by following player 2's strategy.

~~A player X~~ A imagine player 1^x

Suppose B would win by playing 'y'

Instead A plays 'y', and follows

B's strategy

Lemma 2

Player 1 can <u>always</u> get a draw, at least.

Proof

<u>Strategy Stealing</u>: If player 2 has a winning strategy, then player 1 could make an arbitrary first move x, and then win by following player 2's strategy.

~~A plays x_1~~ A imagines playing x_1

Suppose B would win by playing y_1

Instead A plays y_1 and follows B's strategy

Pairing Strategy

$$\begin{bmatrix} 11 & 1 & 8 & 1 & 12 \\ 6 & 2 & 2 & 9 & 10 \\ 3 & 7 & * & 9 & 3 \\ 6 & 7 & 4 & 4 & 10 \\ 12 & 5 & 8 & 5 & 11 \end{bmatrix} \quad n=5$$

Where is there a pairing strategy?

Lemma:

If $n \geq 3^d - 1$ and n is odd or $n \geq 2^d - 1$ and n is even then there is a pairing strategy.

$d=2$ $n=9$ and n odd ✓ $n=5$ ✓

$n=6$ and n even ✓

$n=4$; X can draw

Pairing Strategy

$$\begin{bmatrix} 11 & 1 & 8 & 1 & 12 \\ 6 & 2 & 2 & 9 & 10 \\ 3 & 7 & * & 9 & 3 \\ 6 & 7 & 4 & 4 & 16 \\ 12 & 5 & 8 & 5 & 17 \end{bmatrix} \qquad n=5$$

When is there a pairing strategy?

Lemma 3

If $n \geq 3^d - 1$ and n is odd or if $n \geq 2^d - 1$ and n is even then there is a pairing strategy.

d=2 $\quad n \geq 9$ and n odd ✓ $\qquad n=5$ ✓

$n \geq 6$ and n even ✓

$n = 4$ or 7 ✗ are draws

Generalization of Tic-Tac-Toe.

We have a family $\mathcal{F} = A_1, A_2, \ldots, A_N$, $A_i \subseteq A$

(A_i are the lines).

A move consists of any player choosing unclaimed $x \in A$ and giving it a color.

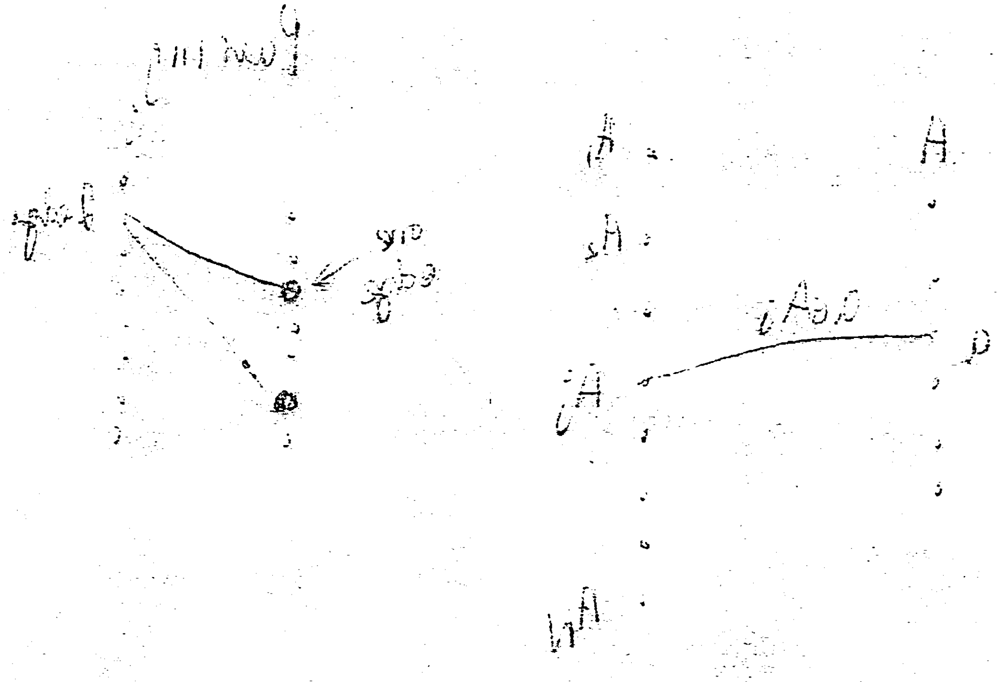
A player wins if he colors a whole A_i

[In Tic-Tac-Toe: $|A_i| = a$, $N = \frac{(a+2)^n - a^n}{2}$]

A winning strategy is a sequence $x_1, x_2, \ldots, x_{2N}$

& there exist a member A_i such that $\{x_{2k+1}\} \supseteq A_i$

for $i = 1, 2, \ldots, N$.



Generalisation of Tic-Tac-Toe.

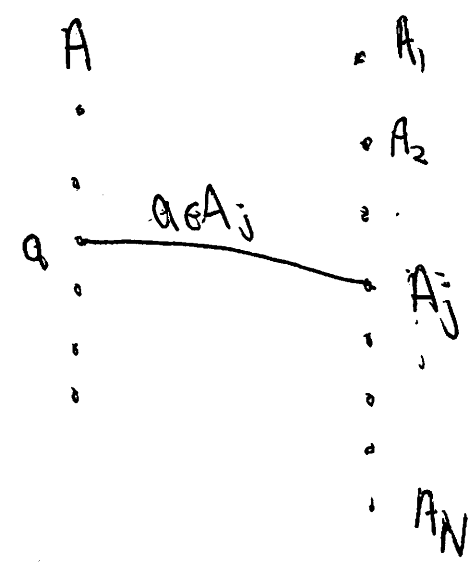
We have a family $\mathcal{F} = A_1, A_2, \dots, A_N \subseteq A$ (A_i corr. to lines).

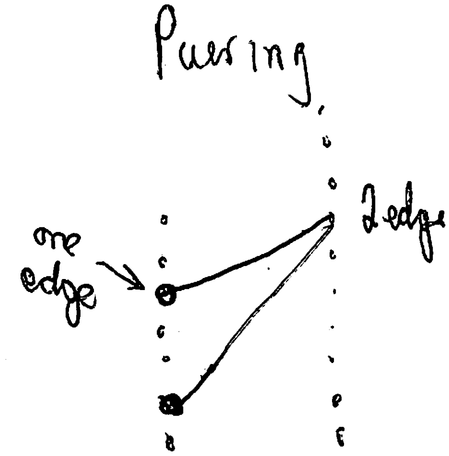
A move consists of one player choosing uncolored $a \in A$ and giving it a color.

A player wins by coloring a whole A_i

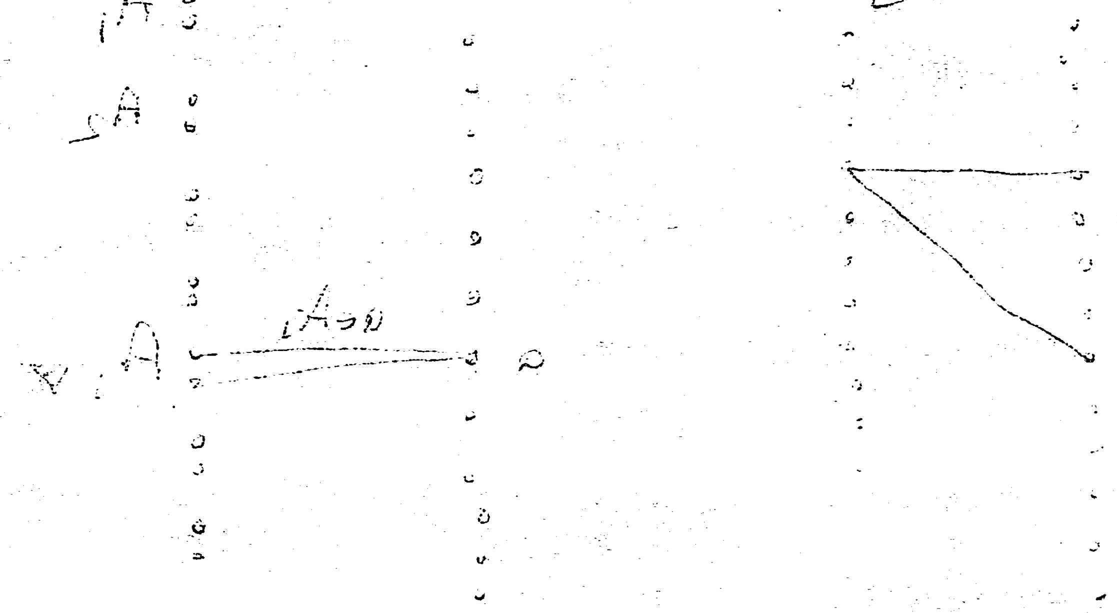
[In Tic Tac Toe : $|A_i| = n$ & $N = \frac{(d+2)^n - d^n}{2}$]

A pairing strategy is a sequence $x_1, x_2, \dots x_{2N}$ of distinct members of A such that $\{x_{2i-1}, x_{2i}\} \subseteq A_i$ for $i = 1, 2, \dots, N$.





Pairing



There is a pairing if there is a matching from RHS into LHS.

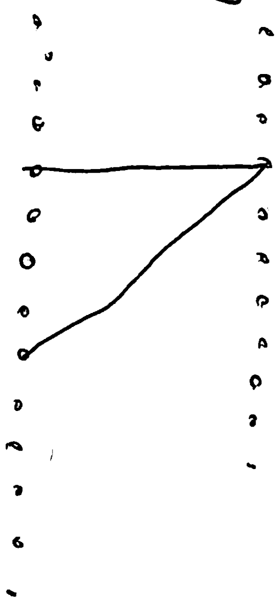
Hall's theorem says that pairing strategy exists iff

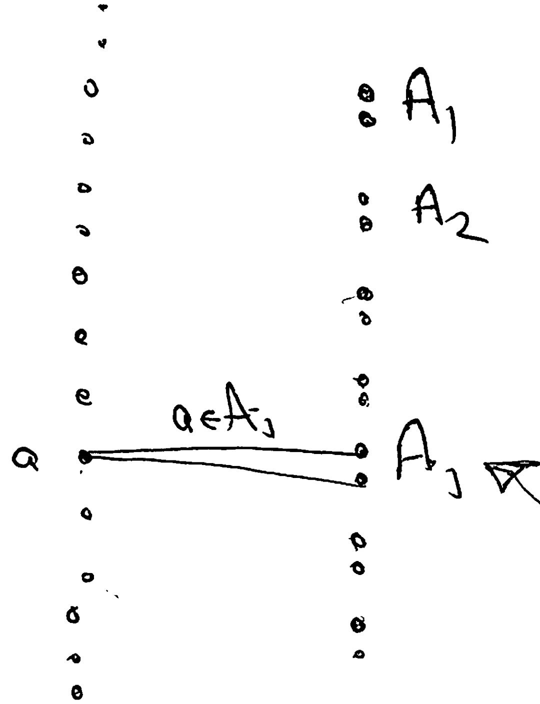
$$\Big|\bigcup_{A \in \mathcal{G}} A\Big| \geq 2|\mathcal{G}|, \ \forall \mathcal{G} \qquad (*)$$

Corollary

If $|A| \geq n$ and if every $x \in A$ is in at most $n/2$ sets of $\mathcal{F}$ then $(*)$ holds.

Pairing





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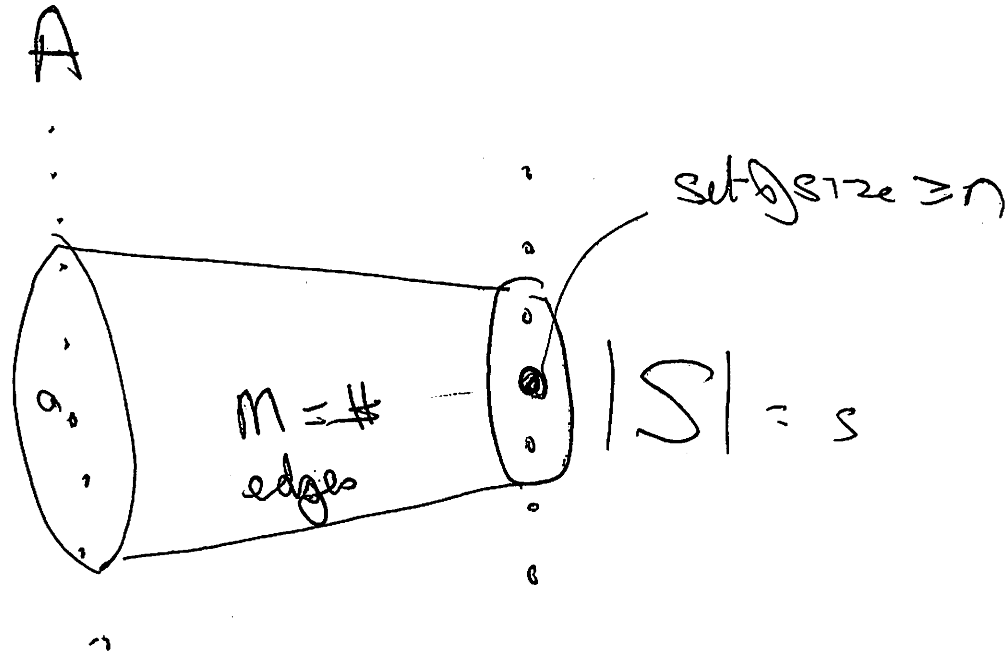
Hall's theorem says that pairing strategy exists iff

$$(*) \quad \Big|\bigcup_{A\in\mathcal{G}} A\Big| \geq 2|\mathcal{G}|, \ \forall\, \mathcal{G}\subseteq\mathcal{F}$$

Corollary

If $|A_i|\geq n$ and if every $a\in A$ is in at most $n/2$ sets of $\mathcal{F}$ then $(*)$ holds.

? $k \geq 2s$



$$sn \leq m \leq \frac{kn}{2}$$