11\11\15

Games $G_1, G_2, \ldots, G_p$

Grundy function $g_1, g_2, \ldots, g_p$

$G = G_1 \oplus G_2 \oplus \cdots \oplus G_p$

Prove: $g = g_1 \oplus g_2 \oplus \cdots \oplus g_p$

Only need to consider $p = 2$ and then use induction.

We have 2 games G_1, G_2 with g_1, g_2

Show that if $G = G_1 \oplus G_2$ then $g = g_1 \oplus g_2$

We can think in terms of the digraph $D = D_1 \times D_2$

D has vertice (x_1, x_2) and edge

$(x_1, x_2) \longrightarrow (x_1, y) \quad y \in N_2^+(x_2)$

$(x_1, x_2) \longrightarrow (y, x_2) \quad y \in N_1^+(x_1)$

Must show:

A1: $x \in X$ & $g(x) = b > a$
$\Rightarrow \exists\, x' \in N^+(x)$ s.t. $g(x') = a$

⋮

Assume $g_1(x_1) \oplus g_2(x_2) = b > a$.

$\cancel{a = d \oplus}$ Let $d = a \oplus b$

$a = d \oplus b = \underline{d \oplus g_1(x_1) \oplus g_2(x_2)}$

Suppose that we can show

(i) $d \oplus g_1(x_1) < g_1(x_1)$ or (ii) $d \oplus g_2(x_2) < g_2(x_2)$

or both

then we will be done. Why

Suppose $d \oplus g_1(x_1) < g_1(x_1)$ then

$\exists\, x_1' \in N_1^+(x_1)$ such that $g_1(x_1') = d \oplus g_1(x_1)$

$$a = g_1(x_1') \oplus g_2(x_2) = g(x_1', x_2)$$

$$\& \quad (x_1', x_2) \in N^+(x_1, x_2)$$

Suppose that $2^{k-1} \leq d < 2^k$

$$d = \frac{1 \; * \; * \; \cdots \; * \; * \; *}{k \qquad\qquad 2 \;\; 1 \;\; 0}$$

$d_k = a_k \oplus b_k$ and $a < b \Rightarrow a_k = 0$ & $b_k = 1$

$$a = d \oplus \underbrace{g_1(x_1) \oplus g_2(x_2)}$$

$a_k = 0$ $d_k = 1$ one of these has 1 in position k

Suppose $g_1(x_1)$ has 1 in position k.

$$d \oplus g_1(x_1) < g_1(x_1)$$

because d "kills" k'th bit and has no effect on higher bits.

Tidying up

To be max we need

(i) $g(x_1', x_2) \neq g(x_1, x_2)$ for $x_1' \in N_1^+(x_1)$

Suppose

$$g_1(x_1') \oplus g_2(x_2) = g_1(x_1) \oplus g_2(x_2)$$

$$\Rightarrow \quad g_1(x_1') = g_1(x_1) \quad \text{— contradiction}$$

(ii) $g_1(x_1) \oplus g_2(x_2) = 0$

$\Rightarrow \quad g_1(x_1') \oplus g_2(x_2) \neq 0$

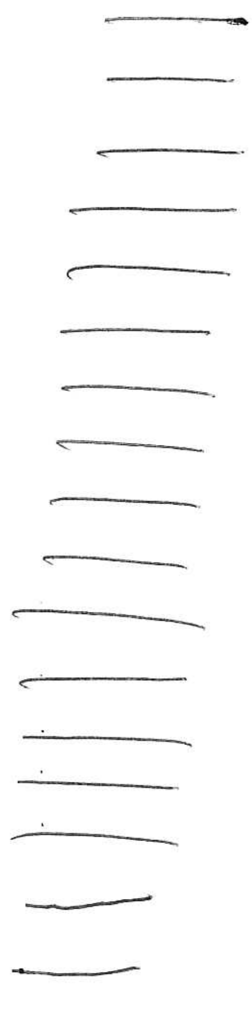
If $\quad g_1(x_1) \oplus g_2(x_2) = 0 = g_1(x_1') \oplus g_2(x_2)$

A more complicated game.

One pile of n chips

First player can remove at most $n-1$ chips

If a player remove x chips the next player remove $\leq x$ chips.



$110 \rightarrow 100$

$1000 \rightarrow 110$

$1000 \rightarrow 100$

1010

n

$$\underbrace{*\,*\,*\,*\,*}_{\text{non-zero}}\,\underbrace{1\,0\,0\,0\,0\,0}$$

Remove if you can

$$*\,*\,*\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0$$

#1's ~~stay~~ same

does not decrease

In general; if a player remove x then next player can remove at most $f(x)$ where $f(x) \geq x$.

(I) There is a set $\mathcal{H} = \{H_1 = 1 < H_2 < \dots\}$ of initial pile sizes are the N positions

If $f(x) = x$ then $H_i = 2^{i-1}$.

(II) $$H_{j+1} = H_j + H_l$$

where $$H_l = \min_{i \leq l} \left(H_i \mid f(H_i) \geq H_l \right)$$

If $f(x) = x$ then $H_j = 2^{j-1}$. — Induction

$$H_{j+1} = 2^{j-1} + \min_{i \leq j} \left[2^{i-1} : 2^{i-1} \geq 2^{j-1} \right]$$

$$= 2^{j-1} + 2^{j-1}$$

$$= 2^j$$

$f(x) = 2$ the $H = \{1, 2, 3, 5, 8, \ldots\}$

$H_j = F_{j-1}$. By induction.

Theorem

Every positive integer can be uniquely written as the sum

$$n = H_{j_1} + H_{j_2} + \cdots + H_{j_p}$$

where $f(H_{j_i}) < H_{j_{i+1}}$.