11\9\15

Game $\equiv$ Digraph - acyclic & finite

Partition vertices in N & P
winning (N), losing (P)

$$x \in N \text{ iff } \underbrace{N^+}_{\text{set of out neighbors}}(x) \cap P \neq \emptyset$$

There is a unique partition that satisfies

Proof is by backwards induction on vertex numbering.

Sums of games

Suppose that we have p games $G_1, G_2, \dots, G_p$ with digraphs $D_i = (X_i, A_i)$, $i = 1, 2, \dots, p$. The sum

$$G_1 \oplus G_2 \oplus \cdots \oplus G_p$$

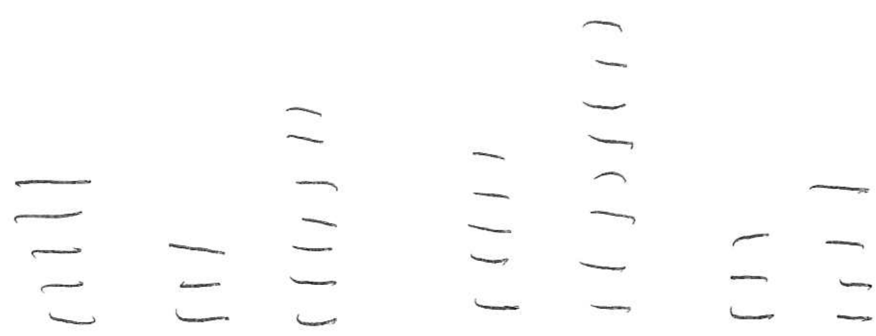
is played as follows

A position is a vector $(x_1, x_2, \dots, x_p) \in X$

$X = X_1 \times X_2 \times X_3 \times \cdots \times X_p$ $\quad$ $\underline{x_i \in X_i}$

To make a move a player choose i such that x_i is not a sink of D_i and then replaces x_i by $y \in N^+(x_i)$.

Nim

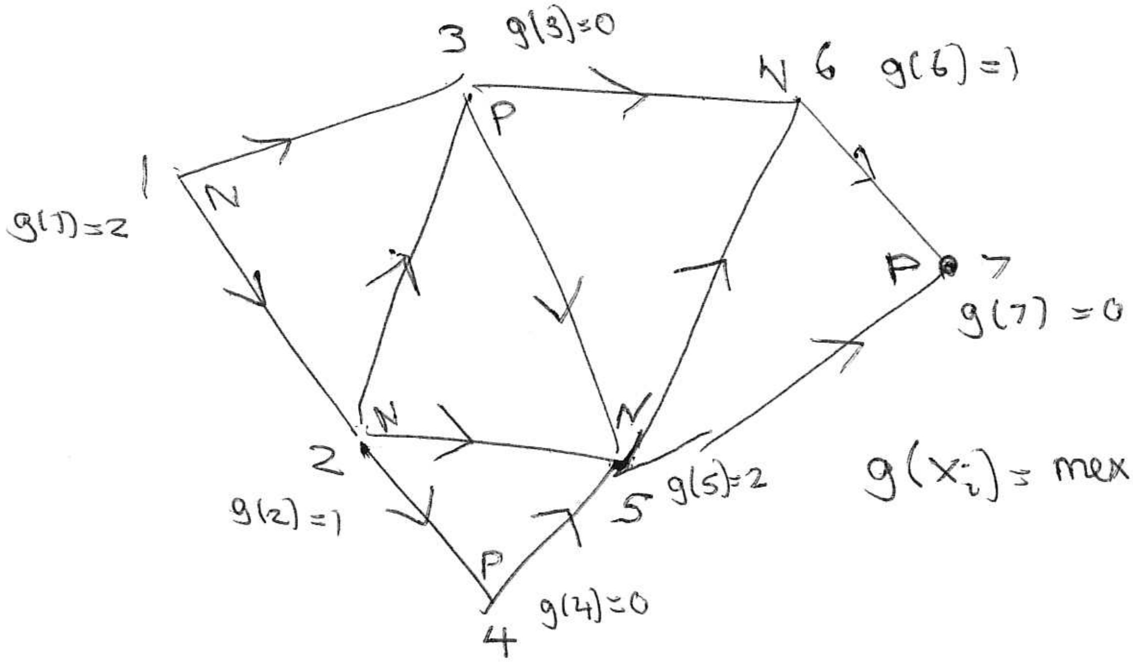


Choose a pile and remove

Sprague - Grundy numbering.

For $S \subseteq \{0, 1, 2, \cdots\}$

$$mex(S) = \min\{x \geq 0 : x \notin S\}$$



$$g(x_i) = mex(N^+(x_i))$$

Given $D = (X, A)$ with ordering $x_1, \ldots, x_n$

we define Grundy number $g : X \to \{0, 1, \ldots\}$

$$x \in P \Leftrightarrow g(x) = 0$$

$$x \in P \Longleftrightarrow \forall\, y \in N^+(x) \Rightarrow \underset{g(y) > 0}{\underline{y \in N}}$$

<u>Main Theorem</u>

$G = G_1 \oplus G_2 \oplus \cdots \oplus G_p$ $\qquad$ G_i has grundy nos g_i

$$g(x_1, x_2, \ldots, x_p) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_p(x_p)$$

$\uparrow$ <u>exclusive or</u> $\qquad$ addition without carry $1+1=0 \ldots$

(1) Take away game $S = \{1, 2, 3, 4\}$

n	0	1	2	3	4	5	6	7	8	9	10
$g(n)$	0	1	2	3	4	0	1	2	3	4	0

$$g(n) = n \bmod 5$$

<u>Proof</u> By induction on n.

$$g(0) = 0$$

Suppose $n = 5r + s$ $\qquad 0 \le s < 5$

$$g(n) = \text{mex}\{0, 1, 2, 3\} = 4 \qquad s = 4$$

$$\text{max}\{0, 1, 2, 4\} = 3 \qquad s = 3$$

$$\text{mex}\{0, 1, 3, 4\} = 2 \qquad s = 2$$

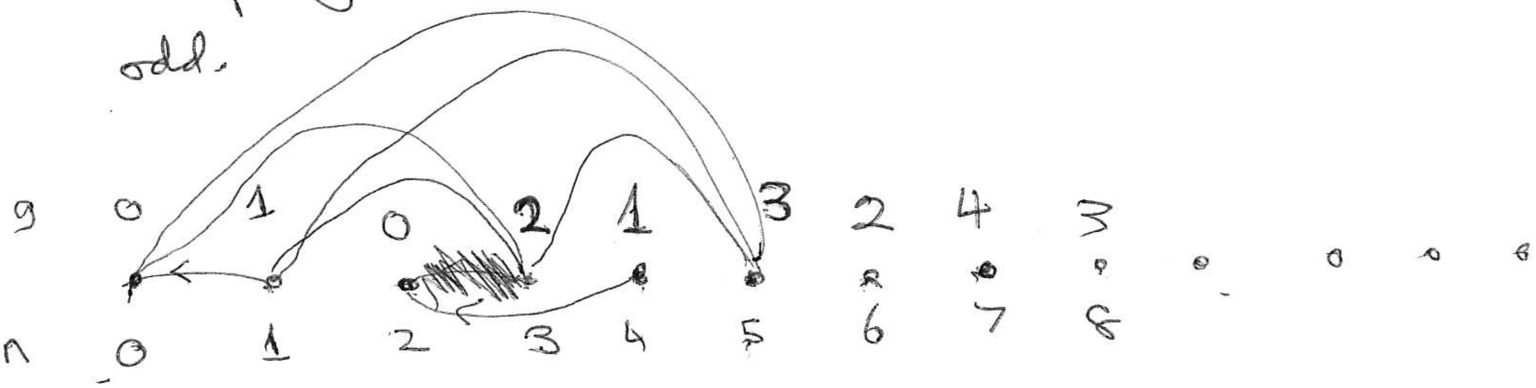
$$\text{mex}\{0, 2, 3, 4\} = 1 \qquad s = 1$$

$$\text{mex}\{1, 2, 3, 4\} = 0 \qquad s = 0$$

Ex 2

A player can remove any even number of chips but not the whole pile.

A player can remove the whole pile if it is odd.



$$g(0) = 0$$

$$g(2k) = k-1 \qquad k \geq 1$$

$$g(2k-1) = k$$

$$g(0) = g(2) = 0, \quad g(1) = 1$$

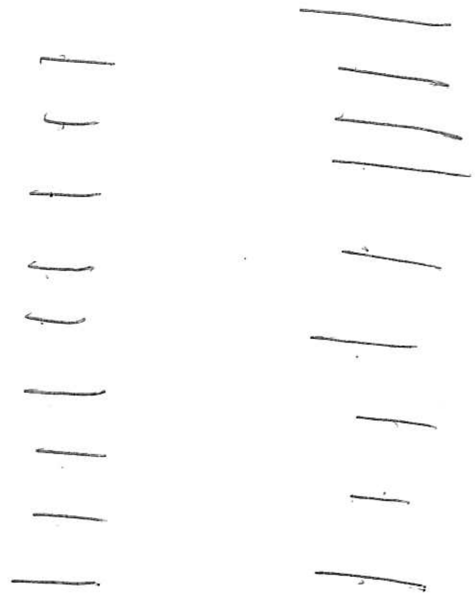
Assume $k > 1$.

$$g(2k) = \operatorname{mex}\{\underset{k-2}{g(2k-2)}, \underset{k-3}{g(2k-4)}, \ldots, \underset{1}{}, \underset{0}{g(2)}\}$$

$$= k-1$$

$$g(2k-1) = \operatorname{mex}\{\underset{k-1}{g(2k-3)}, \underset{k-2}{g(2k-5)}, \ldots, \underset{2}{g(3)}, \underset{1}{g(1)}, \underset{0}{g(0)}\}$$

$$= k$$



Ex 1 Ex 2

$g_1(1) = 4$ $g_2(1) = 5$

$4 \oplus 5 = 1$

$$\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 1 \\ \hline 0 & 0 & 1 \end{array}$$

Winning position.

Winning move: Remove 2 from pile 2, $g_2 \to 4$.