11\6\15

Combinatorial Games

Game 1 a pile

Start with n chips.

Players A, B alternately remove chips from the pile.

A move can remove $s \in S$ chips.

The winner is the player who takes the last chip.

ex
$S = \{1,2,3,4\}$.

$n = 23$

What is the optimal strategy?

Winning Positions $- N$

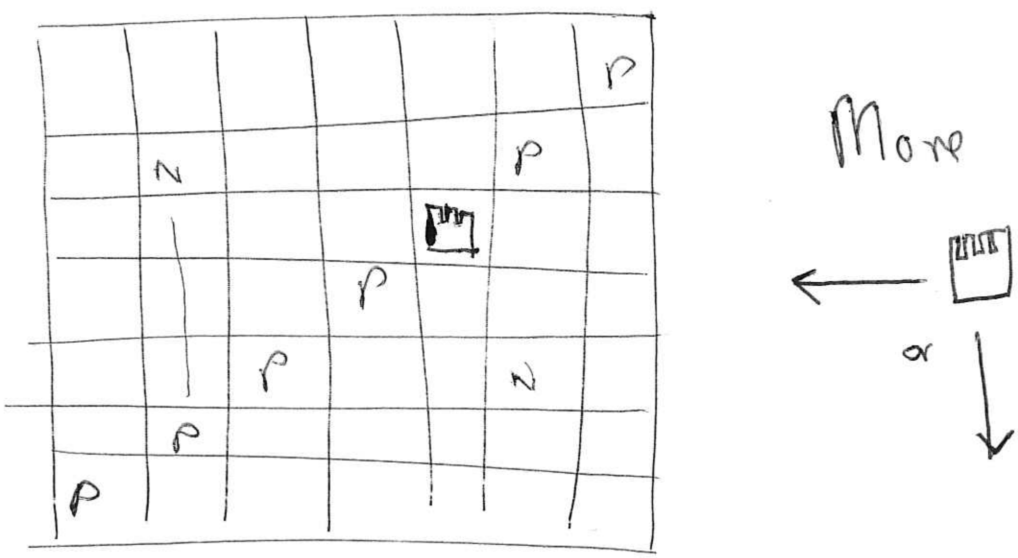
&

Losing Position $- P$ 0, 5, 10, 15, . . .

Variants: (i) Change S

(ii) If a player removes x then next player removes $\leq f(x)$.

Game 2



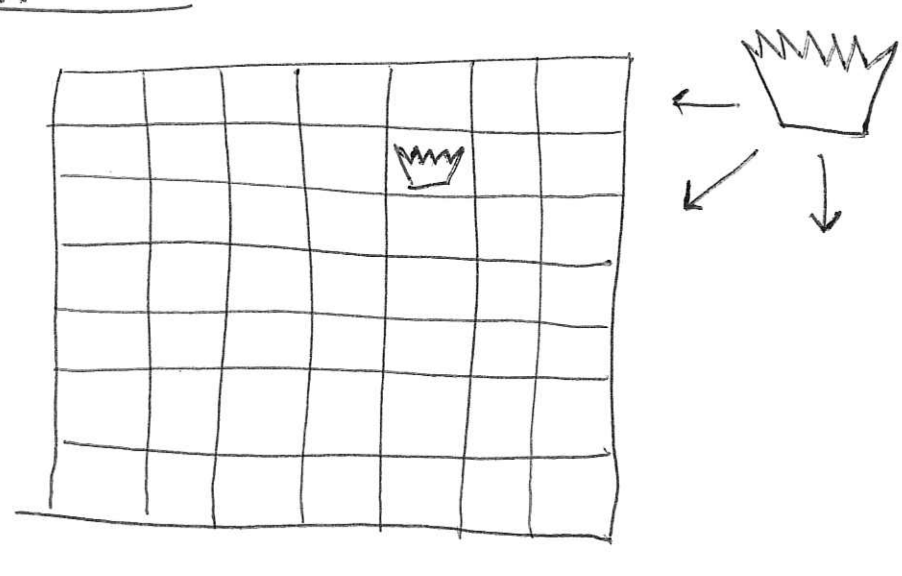
Equivalent to 2-pile nim

Game 3 Wythoff Nim

2 piles: Moves

(i) any number from one pile

(ii) same number from both piles



3

Tic Tac Toe

$n \times n \times n \times \cdots \times n$ game.

4

Geography

W is a set of words.

A & B alternately remove words from W.

Rule is that the first letter of the chosen word = last letter of previous word.

Abstraction

Represent each position of the game by a vertex of a digraph $D = (X, A)$.

(x, y) is an edge if there is a move that takes you from x to y.

We assume that D is finite and is acyclic i.e. no directed cycles.

The game starts with a token on vertex x_0 say and the players alternately move the token to $x_1, x_2, \dots$ where $x_{i+1} \in N^+(x_i)$ — set of out-neighbors.

The game ends at x_m, where $N^+(x_m) = \emptyset$ — sink

Last player to move is the winner.

Ex Game 1

$$V(D) = \{0, 1, 2, \ldots, n\}$$

$(x, y) \in A$ if $x - y \in \{1, 2, 3, 4\}$

Game 2

$$V(D) = \{0, 1, \ldots, m\} \times \{0, 1, \ldots n\}$$

$(x, y) \in N^+((x', y'))$ iff either

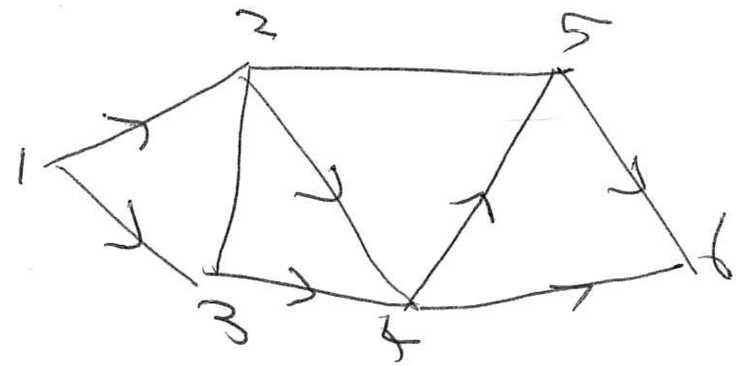
(i) $x = x'$ & $y < y'$

or

(ii) $x < x'$ & $y = y'$

We argue next that such a game must end.

A topological numbering of a digraph $D = (X, A)$ is a map $f: X \to [n]$, $|X| = n$ such that $(x, y) \in A \Rightarrow f(x) < f(y)$.



Thm

A finite digraph D is acyclic iff it has at least one topological numbering

Proof

Suppose there is numbering.

$n_1 < n_2 < n_3 < n_4 < n_1$

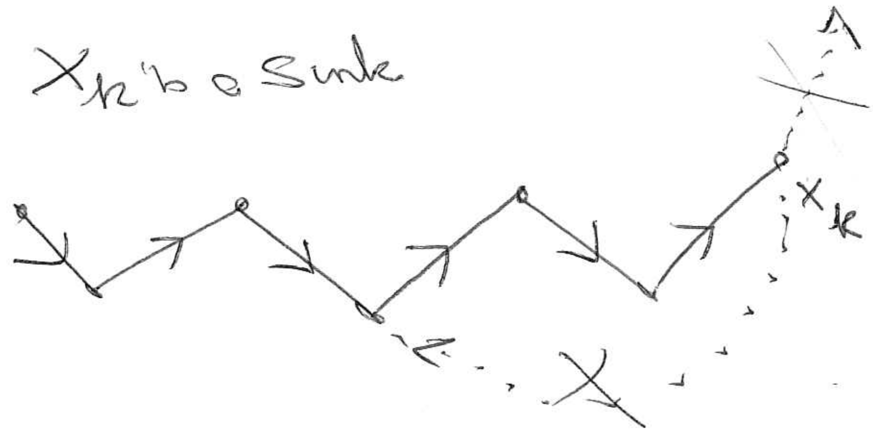
Contradiction

Suppose next that D is acyclic.

We argue first that there is at least one sink

Let $P = (x_1, x_2, \dots, x_k)$ be a longest path

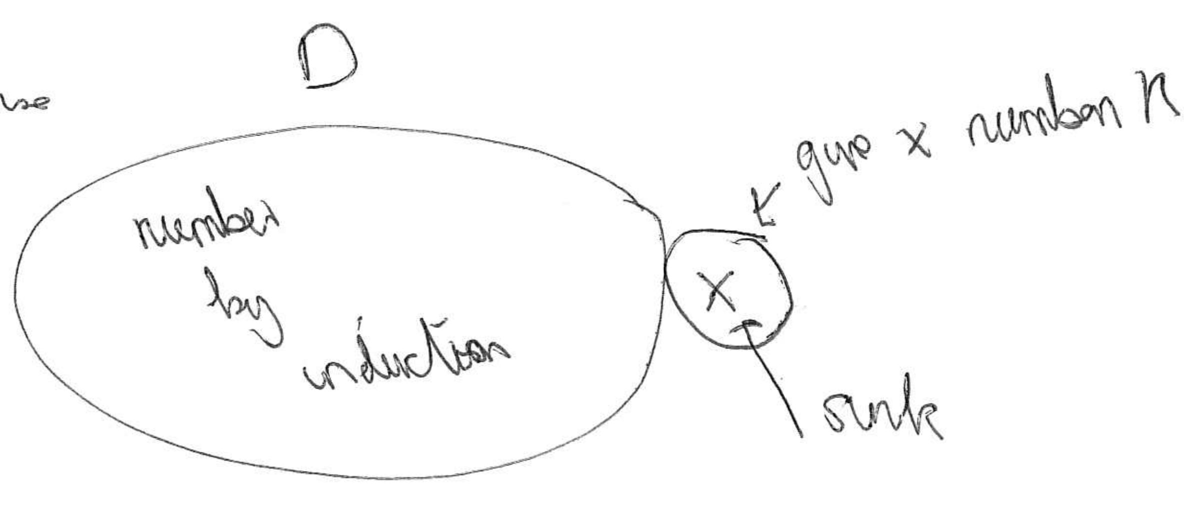
Claim: x_k is a sink



Now we use induction on $|X|$.

If $|X| = 1$ then there is a topological numbering

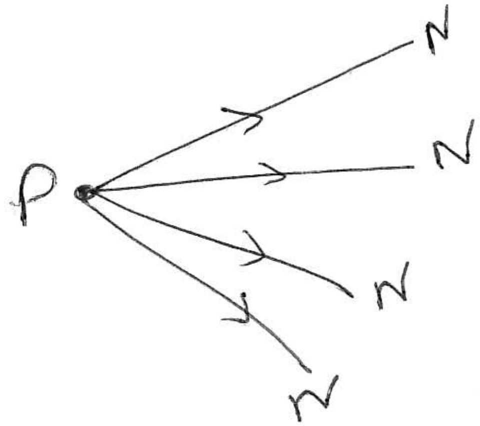
Otherwise

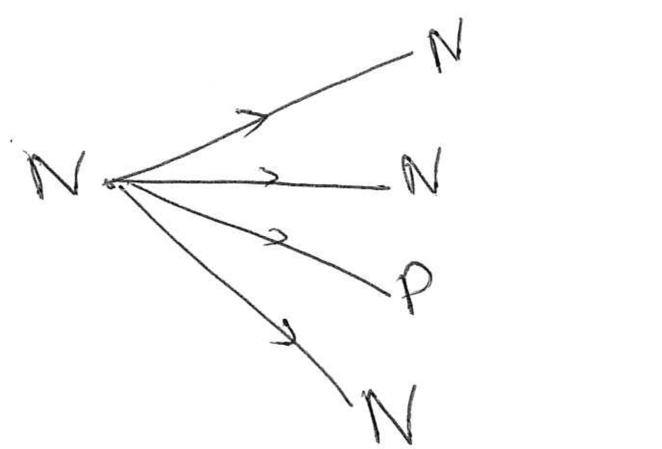


The positions of a game are partitioned into

N (winning positions) — next player will win

P (losing positions) — previous player will win





$$\boxed{x \in N \text{ iff } N^+(x) \cap P \neq \emptyset}$$

Positions of game must alternate   N P N P N $\underline{P}$