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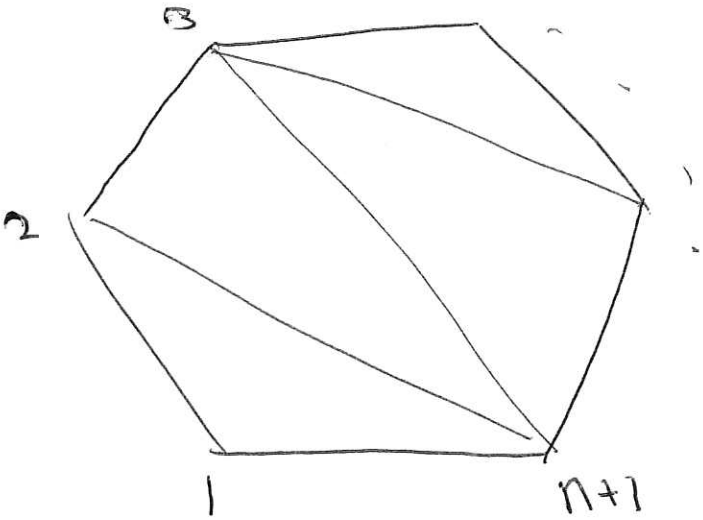
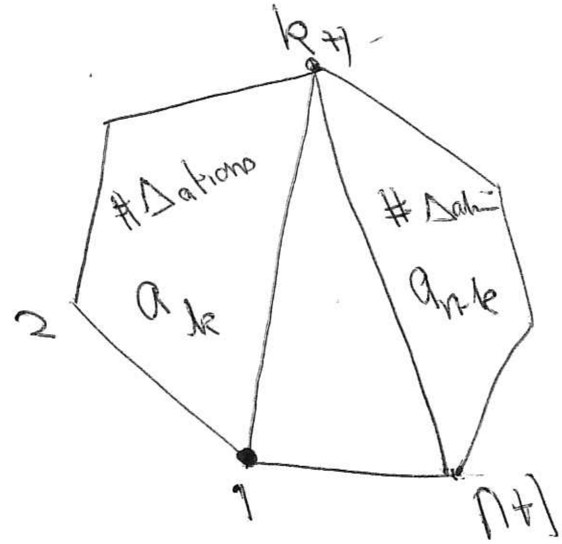


a_n = # of triangulations of (n+1)-gon.

$a_0 = 0$, $a_1 = a_2 = 1$ — $a_3 = 1$ first proper element of sequence.

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$a_n = \sum_{k=0}^{n} a_k a_{n-k}$$



$$\underbrace{\sum_{n=2}^{\infty} a_n x^n}_{a(x)\ -x} = \underbrace{\sum_{\underline{n=2}}^{\infty}\left(\sum_{k=0}^{n} a_k a_{n-k}\right) x^n}_{a(x)^2}$$

$$\downarrow$$

$$\sum_{\underline{n=0}}^{\infty}\left(\sum_{k=0}^{n} a_k a_{n-k}\right) x^n$$

$$\underset{n=0}{a_0^2 = 0} \qquad \underset{n=1}{a_0 a_1 + a_1 a_0 = 0}$$

$$a(x)^2 - a(x) + x = 0$$

$$a(x) = \frac{1+\sqrt{1-4x}}{2} \quad \text{or} \quad \underbrace{\frac{1-\sqrt{1-4x}}{2}}_{a(x)}$$

$$a(0) = a_0 = 0$$

$$a(x) = \frac{1}{2} - \frac{1}{2}(1-4x)^{\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-4x)^n$$

$$a_n = -\frac{1}{2}\binom{\frac{1}{2}}{n}(-4)^n \qquad n \geq 3$$

$$= -\frac{1}{2}\,\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!} \times 4^n \times (-1)^n$$

$$= \cancel{-}\frac{1}{2}\,\frac{\frac{1}{2} \times \cancel{-}\frac{1}{2} \times \cancel{-}\frac{3}{2} \times \cdots \times \cancel{-}\frac{2n-3}{2}}{n!} \times 4^n \times \cancel{(-1)^n}$$

$$= \frac{1 \times 1^{\times 2} \times 3^{\times 4} \times 5 \times \cdots \times 2n-3^{\times 2n-2}}{2^{n+1}\, n! \times 2 \times 4 \times \cdots \times 2n-2} \times 4^n$$

$$= \frac{(2n-2)! \times 4^n}{2^{2n}\, n!\,(n-1)!}$$

$$= \frac{1}{n}\binom{2n-2}{n-1}$$ Catalan Number

Exponential Generating Functions

$a_0, a_1, \ldots a_n, \ldots \longrightarrow$ o.g.f. $\sum_{n=0}^{\infty} a_n x^n$

$\searrow$ e.g.f. $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$

$a_n = n!$ — permutations

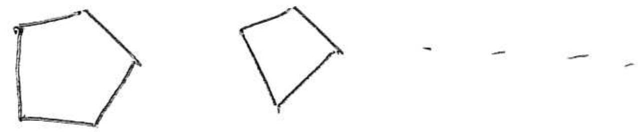
e.g.f. $= \frac{1}{1-x}$

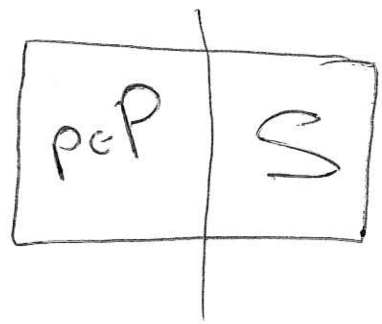
Exponential Families

Example will be the number of permutations consisting of an even number of odd cycles.

P = {pictures}

e.g. a picture could be a cycle

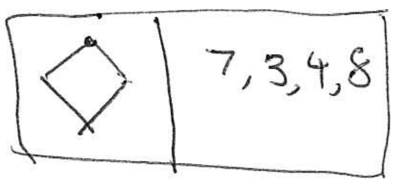


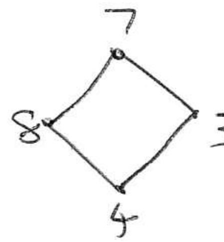
A card C = 

$S = \{\text{labels}\}$

weight of card = $|S|$.

$S = [m]$ - standard card





A hand H is a set of cards whose labels partition $[n]$ for some ~~n~~ $n \geq 1$

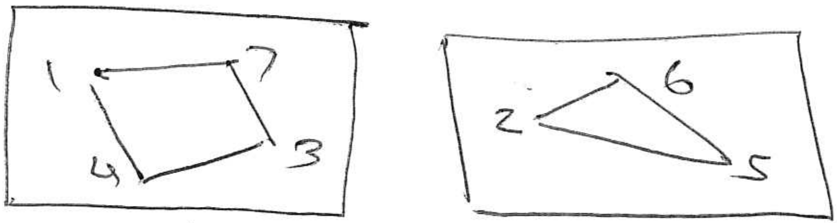
Weight of $H = n$.

$C' = (S', p)$ is a relabelling of (S, p) if $|S'| = |S|$

Deck $\mathcal{D}$ is a finite set of standard cards of common weight n, with distinct pictures.

Exponential family $\mathcal{F} = \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n, \ldots$ ← weight n.

$h(n,k)$ = # of hands of weight n consisting of k cards.



$n=7$
$k=2$

$$H(x,y) = \sum_{\substack{n\geq 1 \\ k\geq 0}} \underline{h(n,k)} \frac{x^n}{n!} y^k$$

Let $d_n = |\mathcal{D}_n|$ = # of choices of pictures of weight n.

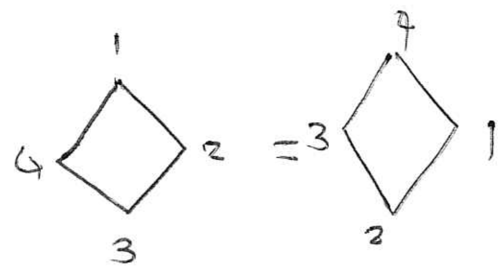
Thm

$$H(x,y) = e^{yD(x)}$$

$$D(x) = \sum_{n=1}^{\infty} \frac{d_n}{n!} x^n$$

$D = \{$ directed cycles of all lengths $\}$

$d_n = (n-1)! = \#$ cyclic permutations



$$D(x) = \sum_{n=1}^{\infty} \frac{(n-1)!\, x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$= \log \frac{1}{1-x}$$

$$H(x,y) = e^{yD(x)} = \exp\left\{ y \log \frac{1}{1-x} \right\} = \frac{1}{(1-x)^y}$$

$\left[n \atop k \right] = \#$ of permutations of $[n]$ with exactly k cycles then

$$\sum_{k=1}^{n} \left[n \atop k \right] y^k = \left[\frac{x^n}{n!} \right] \frac{1}{(1-x)^y} = y(y+1)\cdots(y+n-1)$$