11\2\15

Fibonacci Sequence

$a_0 = a_1 = 1$

$$a_n - a_{n-1} - a_{n-2} = 0 \qquad n \geq 2$$

$$\sum_{n=2}^{\infty} (a_n - a_{n-1} - a_{n-2}) x^n = 0$$

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$[a(x) - a_0 - a_1 x] - [x(a(x) - a_0)] - x^2 a(x) = 0$$

$$a(x) = \frac{1}{1 - x - x^2} \qquad \xi_1 = -\frac{\sqrt{5}+1}{2}$$

$$= \frac{1}{(\xi_1 - x)(\xi_2 - x)} \qquad \xi_2 = \frac{\sqrt{5}-1}{2}$$

$$= \frac{1}{\xi_1 - \xi_2}\left(\frac{\xi_1^{-1}}{1 - x/\xi_1} - \frac{\xi_2^{-1}}{1 - x/\xi_2} \right)$$

$$= \frac{\xi_1^{-1}}{\xi_1 - \xi_2} \sum_{n=0}^{\infty} \xi_1^{-n} x^n - \frac{\xi_2^{-1}}{\xi_1 - \xi_2} \sum_{n=0}^{\infty} \xi_2^{-n} x^n$$

$$a(x) = \sum_{n=0}^{\infty} \underbrace{\frac{\xi_1^{-n-1} - \xi_2^{-n-1}}{\xi_1 - \xi_2}}_{a_n} x^n$$

$$\xi_1 = -\frac{\sqrt{5}+1}{2}$$

$$\xi_2 = \frac{\sqrt{5}-1}{2}$$

Another example - inhomogeneous

$$a_n - 3a_{n-1} = n^2, n \geq 1 . a_0 = 1$$

$$\underbrace{\sum_{n=1}^{\infty} (a_n - 3a_{n-1}) x^n}_{[a(x) - a_0] - 3x\, a(x)} = \sum_{n=1}^{\infty} n^2 x^n$$

$$[a(x) - a_0] - 3x\, a(x) = \sum_{n=1}^{\infty} n(n-1) x^n + \sum_{n=1}^{\infty} n x^n$$

$$a(x)(1 - 3x) - 1 = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

$$= \frac{x + x^2}{(1-x)^3}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + n x^{n-1}$$

$$\frac{2}{(1-x)^3} = 2 + 2 \times 3 x + \cdots\cdots n(n-1) x^{n-2}$$

$$a(x) = \frac{x + x^2}{(1-x)^3(1-3x)} + \frac{1}{1-3x}$$

$$= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{1-3x}$$

$A = -\frac{1}{2}$, $B = 0$, $C = -1$, $D = 3/2$

$$= -\frac{1}{2}\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} \binom{n+2}{2} x^n + \frac{5}{2}\sum_{n=0}^{\infty} 3^n x^n$$

$$a_n = -\frac{1}{2} - \binom{n+2}{2} + \frac{5}{2} 3^n$$

Products of Generating Functions

$$a(x) = \sum_{n=0}^{\infty} a_n x^n \quad \& \quad b(x) = \sum_{n=0}^{\infty} b_n x^n$$

$$a(x)\,b(x) = (a_0 + a_1 x + a_2 x^2 + \cdots) \times (b_0 + b_1 x + b_2 x^2 + \cdots)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$$

$$= \sum_{n=0}^{\infty} c_n x^n$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Derangements

$d_n = \#$ derangements

$$1\ \cancel{n!} = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}$$

fixed pts $\pi(i)=i$

choose k fixed points

fill in rest with a derangement

$$= \cancel{n!} \sum_{k=0}^{n} \frac{d_{n-k}}{(n-k)!} \cdot \frac{1}{k!}$$

Let $d(x) = \sum_{m=0}^{\infty} \frac{d_m}{m!} x^m$

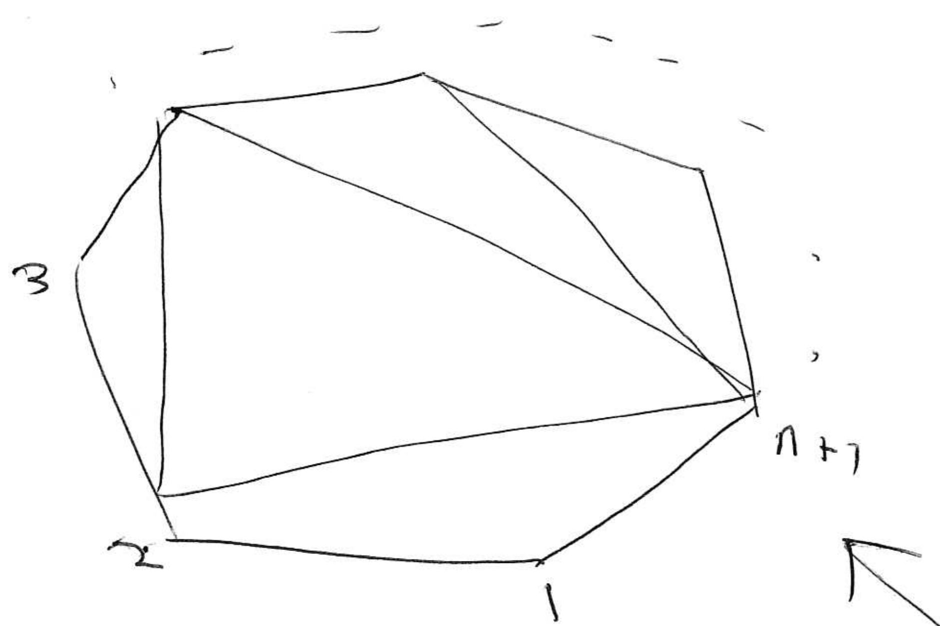
$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{d_{n-k}}{(n-k)!} x^{n-k} \cdot \frac{x^k}{k!}$$

$$\frac{1}{1-x} = d(x) \cdot e^x$$

So $d(x) = \dfrac{e^{-x}}{1-x}$

$$d(x) = \frac{e^{-x}}{1-x}$$

$$\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$



$a_n =$ # of triangulations of

$a_0 = 0,\ a_1 = a_2 = 1,\ a_3 = 1$ $\qquad a(x) = \sum_{n=0}^{\infty} a_n x^n$

① $a_n = \sum_{k=0}^{n} a_k a_{n-k} \longrightarrow a(x) = x + a(x)^2$

$n \geq 2$

$\Downarrow$