10/7/15

Ramsey's Theorem

Given positive integers $k, l \geq 1$, there exists $R(k,l)$ such that if $n \geq R(k,l)$ then in any two coloring (Red/Blue) of the edges of K_n, there exists either a copy of K_k with all Red edges or a copy of K_l, ~~all~~ with all blue edges.

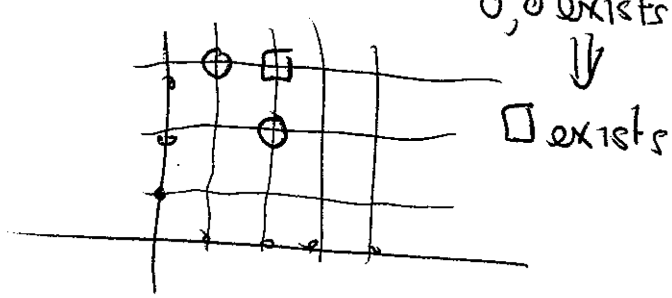
<u>Proof</u>

We know that $R(1,k)$, $R(2,k)$, $R(k,1)$, $R(k,2)$ exist for all k.

We show that

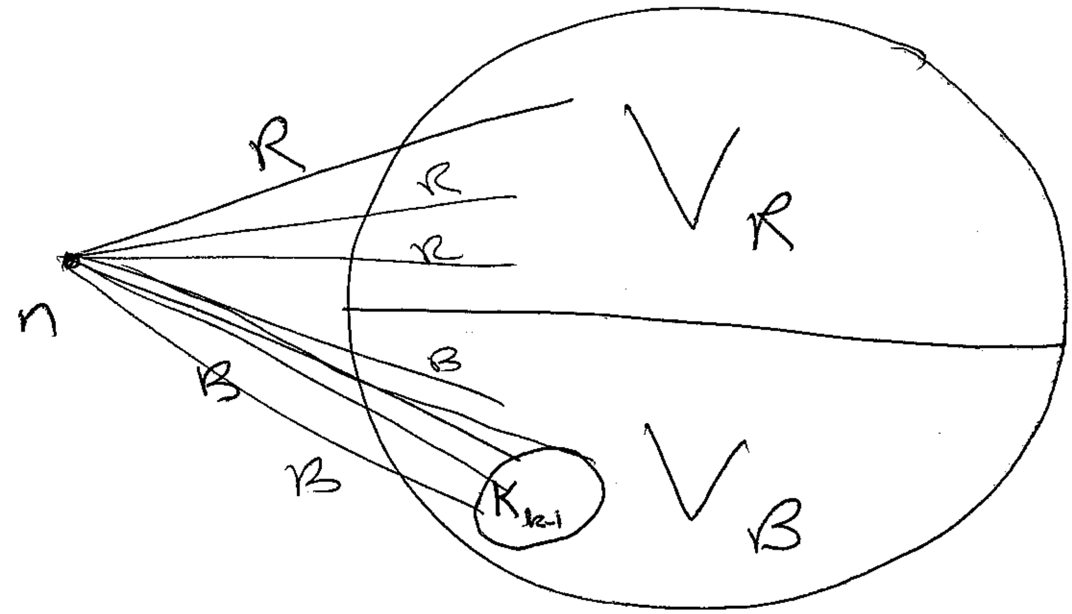
$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$

This shows that $R(k,l)$ exists for all k,l, by induction on $k+l$.



Assume inductively that $R(k-1,l)$ & $R(k,l-1)$ exist.
Let: $n = R(k-1,l) + R(k,l-1)$.

Take any 2-coloring of K_n.



$|V_R| + |V_B| = n-1$

Either (i) $|V_R| \geq R(k-1,l)$

or (ii) $|V_B| \geq R(k,l-1)$

Assume (i) is true.

Then we a 2-coloring of the edges of V_R.

Either $\exists$ Blue $K_l \subseteq V_R$ ✓

or $\exists$ Red $K_{k-1} \subseteq V_R$. Add n to get a Red K_k

$$R(k,l) \leq \binom{k+l-2}{k-1}$$ ~~for k+l ≥~~

Proof

Induction on $k+l$. True for $k+l$ small.

$k=1$, $l \geq 1$: $R(1,l) = 1$

$k \geq 1$, $l = 1$

$k+l=2$ as a base case

$k=2$, $l \leq 2$: $R(2,2) = 2$

Inductive Step:

$$R(k,l) \leq R(k-1,l) + R(k,l-1)$$

$$\leq \binom{k+l-3}{k-2} + \binom{k+l-3}{k-1}$$

$$= \binom{k+l-2}{k-1}.$$

In particular $R(k,k) \leq \binom{2k-2}{k-1} < 4^k$.

Lower bound.

$$R(k,k) > 2^{k/2}$$

We show that if $n \leq 2^{k/2}$ then there is a 2-coloring of K_n with no Red K_k or Blue K_k.

We take a random 2-coloring of K_n.

$C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$ are the vertices of the k-cliques of K_n.

$\mathcal{E}_{R,j}$ = C_j is Red

$\mathcal{E}_j = \mathcal{E}_{R,j} \vee \mathcal{E}_{B,j}$

$\mathcal{E}_{B,j}$ = C_j is Blue

$\mathcal{E}_R = \exists j : \mathcal{E}_{R,j}$ $\quad$ $\mathcal{E}_B = \exists j : \mathcal{E}_{B,j}$

$$Pr(\mathcal{E}_R \vee \mathcal{E}_B) \leq P(\mathcal{E}_R) + P(\mathcal{E}_B)$$

$$= 2P(\mathcal{E}_R)$$

$$= 2P\left(\bigcup_{j=1}^{N} \mathcal{E}_{R,j}\right)$$

$$\leq 2\sum_{j=1}^{N} P(\mathcal{E}_{R,j})$$

$$= 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}}$$

$$\leq 2\,\frac{n^k}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}}$$

$$\leq 2\,\frac{2^{k^2/2 - \binom{k}{2}}}{k!}$$

$$= \frac{2^{1+k/2}}{k!}$$

$$< 1$$

$R(3,3) = 6$

$R(3,4) = 9$

$R(4,4) = 18$

$R(4,5) = 25$

$43 \leq R(5,5) \leq 49$

More general version

Ramsey's theorem is about 2-colorings of the 2-element subsets of $[n]$

(1) We can use more colors

(2) We can color the r-subsets of $[n]$

Theorem

Let $r, s \geq 1$, $q_1, q_2, \ldots, q_s \geq r$ be given

set size # colors

$\exists\, N = N(q_1, q_2, \ldots, q_s; r)$ such that if $m \geq N$ and we color all the r-subsets of $[m]$ with r colors then $\exists\, 1 \leq i \leq s$ and a set $T \subseteq [m]$ such that $|T| = q_i$ and all r-subsets of T are given color i.