10\5\15

The Pigeon Hole Principle PHP

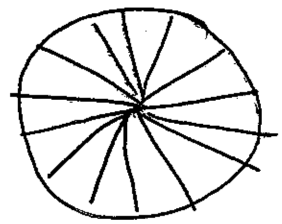
Ex 1: $A \subseteq [2n]$, $|A| = n+1$ then A contains a pair x, y such that x & y are co-prime.

$A = \{2, 4, 6, \dots, 2n\} \Rightarrow$ can't replace $n+1$ by n.

Pf $|A| = n+1$ then PHP $\Rightarrow \exists$ a pair $x, x+1 \in A$

Ex2

We have 2 disks, each partitioned into 200 equal size sectors.



Disk 1 has 100 Red Sectors and 100 Blue Sectors. [Arbitrary coloring]

Disk 2 is colored arbitrarily.

Claim: There's a way of placing Disk 2 onto Disk 1 so that ≥ 100 sectors of Disk 2 have the same color sector below them.

Fix Disk 1 in position.

There are 200 positions for Disk 2.

Let q_i = # of sectors of Disk 2 with same color underneath.

Claim: $q_1 + q_2 + \cdots + q_{200} = 200 \times 100$

$\Rightarrow$ Result.

Consider 200×200 matrix A.

$$A(i,j) = \begin{cases} 1 & \text{sector } j \text{ is "good" in position } i \\ 0 & \text{otherwise} \end{cases}$$

sector $\to j$

position $\to i$

$$A \quad i \left[\quad 1_{\text{sector } j \text{ is good}} \quad \right] \leftarrow \text{row sum} = q_i$$

100 in each column

$$\# \text{ of } 1\text{s} = q_1 + q_2 + \cdots + q_{200} \quad \text{(Row Sum)}$$

$$= 200 \times 100 \quad \text{(Column Sum)}$$

Alternative Solution.

Place Disk 2 in a random position.

$$X_i = \begin{cases} 1 & \text{sector } i \text{ of Disk 2 is good} \\ 0 & \text{otherwise.} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_{200} = \#\text{good sectors}$$

$$E(X_i) = \tfrac{1}{2}. \qquad EX = 100$$

$$\Rightarrow \exists \text{ a position}$$

(Erdös-Szekores) An arbitrary sequence of integers $a_1, a_2, \cdots, a_{k^2+1}$ contains a monotone subsequence of length $k+1$.

($i_1, i_2, \ldots, i_\ell$ is monotone if either

$$a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_\ell}$$

or $$a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_\ell}$$

Proof

Let $(a_i, a_i^1, a_i^2, \cdots, a_i^{\ell-1})$ be the longest monotone increasing subsequence that starts with a_i, $(1 \leq i \leq k^2+1)$ and let $\ell(a_i)$ be its length.

$$3, \quad 5, \quad 2, \quad 7, \quad 4, \quad 12$$

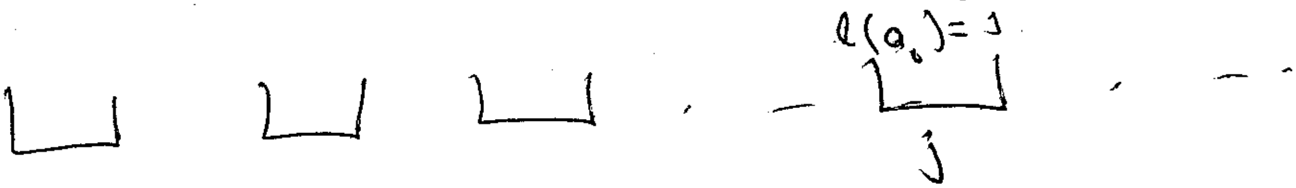
$\ell(1) = 4 \quad \ell(2) = 3 \quad \ldots$

Case 1: $\exists\, i$ such that $l(a_i) \geq k+1$.

Done.

Case 2: $l(a_i) \leq k$ for all i.

If $l(a_i) = j$, put a_i into box j



There is a box with at least $k+1$ elements.

$$\cdots \quad \boxed{i_1, i_2, \dots, i_{k+1}} \quad \cdots$$

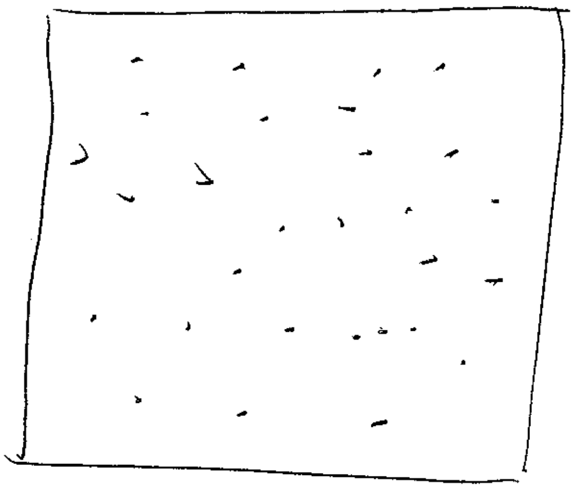
Claim: $a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_{k+1}}$ | $i_1 < i_2 < \cdots$

Suppose $a_{i_t} < a_{i_{t+1}}$

Contradiction $l(a_{i_t}) \geq l(a_{i_{t+1}}) + 1$.

The sequence

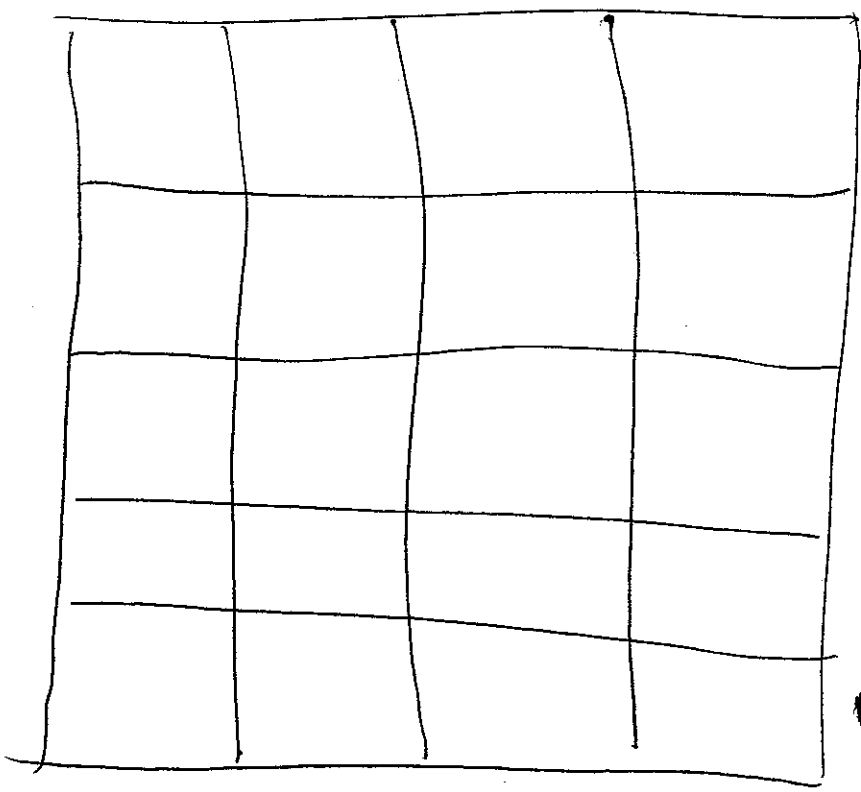
$$n, n-1, \dots, \underline{1}, \ 2n, 2n-1, \dots, n+1, \dots\dots, n^2, n^2-1, \dots$$



Let $P_1, P_2, \dots, P_n$ be n points in $[0,1]^2$

$\exists\, i, j, k$ such that area $\triangle P_i P_j P_k$

$$\le \frac{1}{2\left(\lfloor\sqrt{(n-1)/2}\rfloor\right)^2} \approx 1/n$$



$m = \lfloor \sqrt{(n-1)/2} \rfloor$

Divide $[0,1]^2$ into $m^2 < \frac{n}{2}$ subsquares.

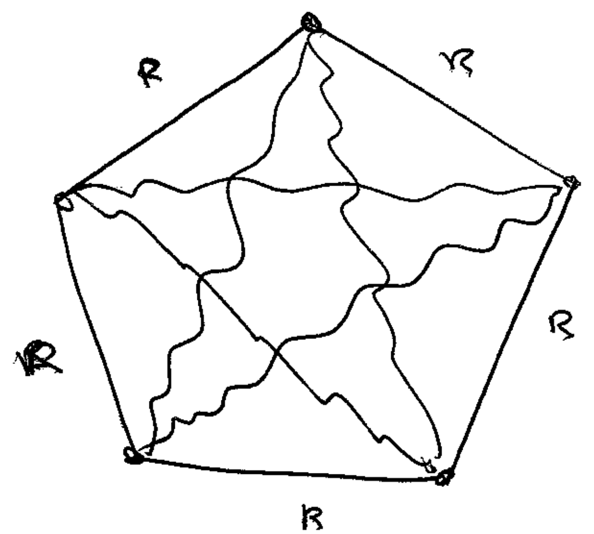
Some square contains ≥ 3 points P_i, P_j, P_k

Area of $\triangle \le \frac{1}{2}$ area of of a subsquare

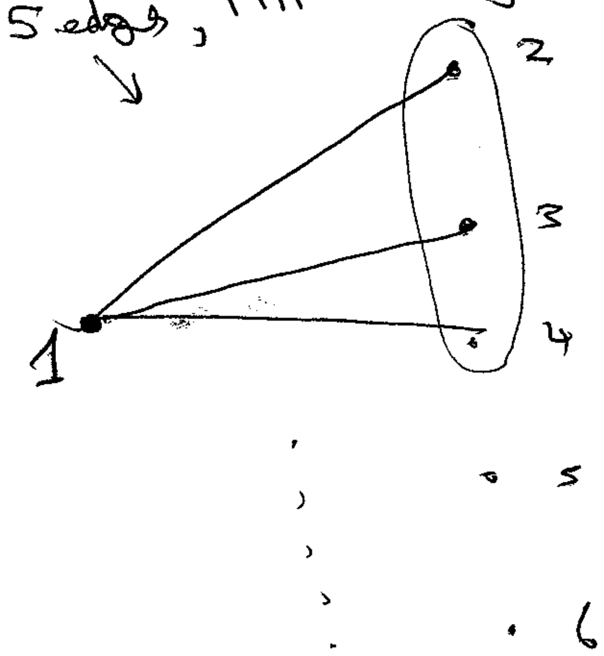
Ramsey Theory

Suppose we two color the edges of K_6, Red & Blue. Then there must be either a Red $\triangle$ or a Blue $\triangle$.

Doesn't work for K_5



5 edges, PHP $\Rightarrow$ 3 same color.



$\xrightarrow{R}$

B

If Δ 234 has a Red edge - done

Otherwise it is Blue - done.

Generalise: $R(s,s)$ = minimum integer such that if $n \geq R(s,t)$

then in any 2-coloring of the edge of K_n there is a Red K_s, ~~all of whose edges have same~~ or a Blue K_t.

We just showed that $R(3,3) = 6$

$R(4,4) = 18$

$R(1,k) = R(k,1) = 1$

$R(2,k) = R(k,2) = k$

$R(5,5) = ??$