10/30/15

Sequence $a_0, a_1, \ldots, a_n, \ldots$ is an infinite sequence.

A recurrence relation is a set of equations

$$a_n = f_n(a_{n-1}, a_{n-2}, \ldots, a_{n-k}).$$

The whole sequence is determined by $a_0, a_1, \ldots, a_{k-1}$.

~~Linear~~ Recurrence

Fibonacci Sequence

$$a_n = a_{n-1} + a_{n-2}$$

$$a_0 = a_1$$

1, 1, 2, 3, 5, 8, 13, 21, ...

$$b_n = |B_n| = |\{x \in \{a,b,c\}^n : aa \text{ does not appear in } x\}|$$

$b_1 = 3$: a b c

$b_2 = 8$ ab ac ba bb bc ca cb cc

<u>Claim</u>

$$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 2$$

$$B_n = B_n^{(b)} \cup B_n^{(c)} \cup B_n^{(a)}$$

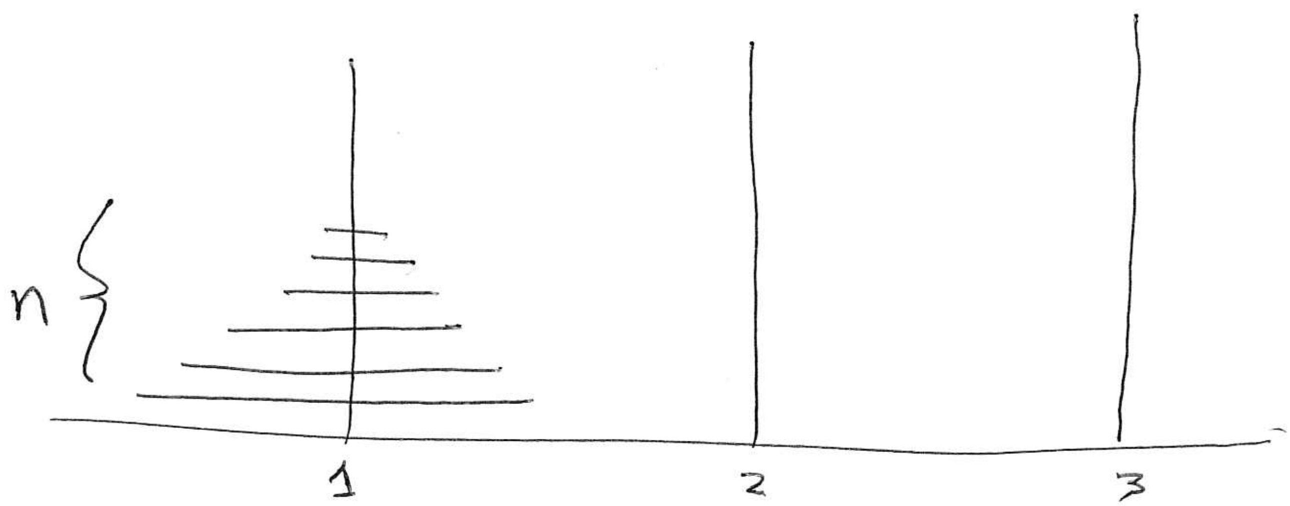
where $B_n^{(\alpha)} = \{x \in B_n : x_1 = \alpha\}$, $\alpha = a, b, c$.

$$|B_n^{(b)}| = |B_n^{(c)}| = b_{n-1}$$

$$B_n^{(a)} = B_n^{(ab)} \cup B_n^{(ac)}$$

& $|B_n^{(ab)}| = b_{n-2}$

Towers of Hanoi



Move rings from 1 to 3

H_n = minimum number of moves.

$H_1 = 1$

$$H_n = H_{n-1} + 1 + H_{n-1}$$

$$= 2H_{n-1} + 1$$

$n=$	1	2	3	4	
	1,	3,	7,	15	...

$H_n = 2^n - 1$

Proof by induction

Ordinary Generating Functions

Sequence

$$a_0, a_1, a_2, \dots\dots a_n, \dots$$

$$\updownarrow$$

$$a(x) = a_0 + a_1 x + a_2 x^2 + \dots\dots$$

We write

$$a_n = [x^n]\, a(x)$$

$a_n = 1$:

$$a(x) = 1 + x + x^2 + \dots = \frac{1}{1-x}$$

$G_n = n+1$

$$a(x) = 1 + 2x + 3x^2 + \dots = \frac{1}{(1-x)^2}$$

$a_n = n$

$$a(x) = x + 2x^2 + 3x^3 + \dots = \frac{x}{(1-x)^2}$$

Newton's Binomial Theorem

$$a_n = \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$. α is Real or Complex

$$a(x) = \underbrace{(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n}_{\text{Calculus}}$$

Taylor Series

$$a(x) = \frac{1}{(1-x)^m} = \sum_{n=0}^{\infty} \binom{-m}{n} (-x)^n$$

$$= \sum_{n=0}^{\infty} \binom{m+n-1}{m} x^n$$

$$\binom{-m}{n} = \frac{(-m)(-m-1)\cdots(-m-n+1)}{n!}$$

$$= (-1)^n \frac{m(m+1)\cdots(m+n-1)}{n!}$$

$$= (-1)^n \binom{m+n-1}{m}$$

Solution of linear Recurrence

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

$a_0 = 1$, $a_1 = 9$

$$\sum_{n=2}^{\infty} (a_n - 6a_{n-1} + 9a_{n-2}) x^n = 0$$

$$\sum_{n=2}^{\infty} a_n x^n = a(x) - a_0 - a_1 x = a(x) - 1 - 9x$$

$$\sum_{n=2}^{\infty} a_{n-1} x^n = x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} = x(a(x) - a_0)$$
$$= x(a(x) - 1)$$

$$\sum_{n=2}^{\infty} a_{n-2} x^n = x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = x^2 a(x).$$

$$a(x) - 1 - 9x - 6x(a(x) - 1) + 9x^2 a(x) = 0$$

$$a(x) = \frac{1+3x}{1-6x+9x^2} = \frac{1+3x}{(1-3x)^2}$$

$$a(x) = \frac{1}{(1-3x)^2} + \frac{3x}{(1-3x)^2}$$

$$= \sum_{n=0}^{\infty} (n+1)(3x)^n + 3x \sum_{n=0}^{\infty} (n+1) 3^n x^n$$

$$= \sum_{n=0}^{\infty} (n+1) 3^n x^n + \sum_{n=0}^{\infty} (n+1) (3x)^{n+1}$$

$$= \sum_{n=0}^{\infty} (n+1) 3^n x^n + \sum_{n=0}^{\infty} n 3^n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+1) 3^n + n 3^n \right] x^n$$

$$a_n = (2n+1) 3^n$$