10\2\15

Erdös-Ko-Rado theorem.

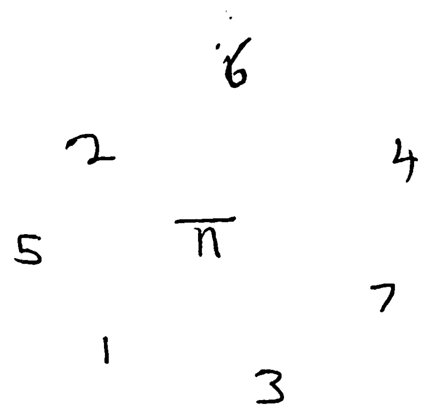
Theorem

If $\mathcal{A}$ is an intersecting family and $\mathcal{A} \subseteq \binom{[n]}{k}$, $k \le \lfloor n/2 \rfloor$, then $|\mathcal{A}| \le \binom{n-1}{k-1}$.

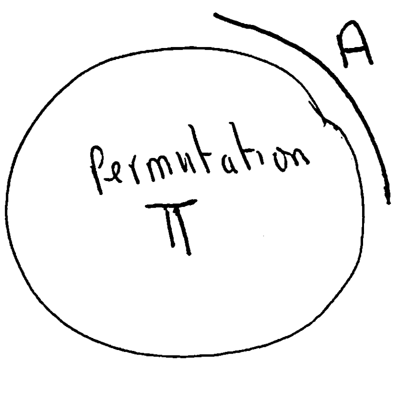
[tight can take all k-sets containing element 1]

Proof

Circular permutation



$\theta(\pi, \{4,6,7\}) = 1$



$\theta(\pi, A) = 1$

Claim: $\displaystyle\sum_{A \in \mathcal{A}} \theta(\pi, A) \le k, \quad \forall \pi$

Now suppose that π is a random permutation

$$E(\theta(\pi, A)) = \frac{\underbrace{n}_{\text{beginning of interval}} \times \underbrace{k!}_{\text{order for } A} \times \underbrace{(n-k)!}_{\text{order the rest}}}{n!} = \frac{k}{\binom{n-1}{k-1}}$$

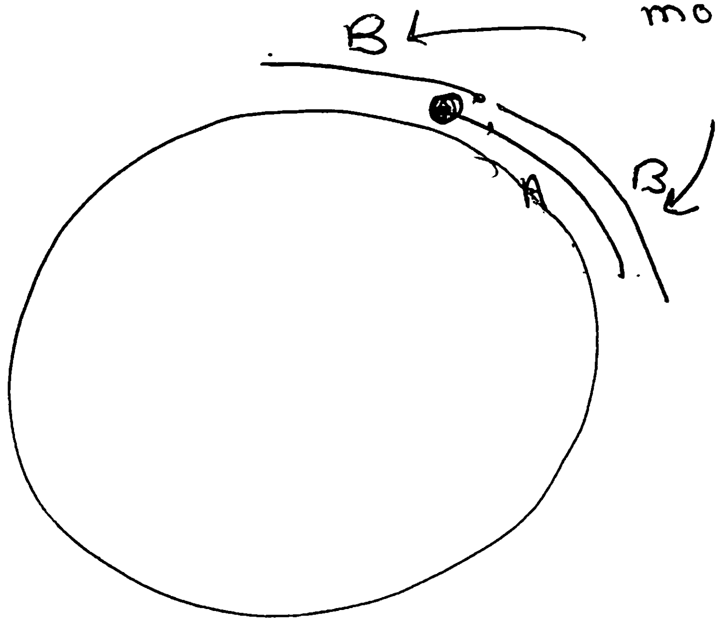
$$k \geq E\Big(\sum_{A \in \mathcal{A}} \theta(\pi, A)\Big) = \sum_{A \in \mathcal{A}} \frac{k}{\binom{n-1}{k-1}}$$

$$= |\mathcal{A}| \times \frac{k}{\binom{n-1}{k-1}}$$

$$\Rightarrow |\mathcal{A}| \leq \binom{n-1}{k-1}.$$

<u>Claim</u>

$B \in \mathcal{A}$ can be in at most one of



Now suppose that π is a random permutation.

$$E(\theta(\pi, A)) = \frac{n \times k! \times (n-k)!}{n!} = \frac{k}{\binom{n-1}{k-1}}$$

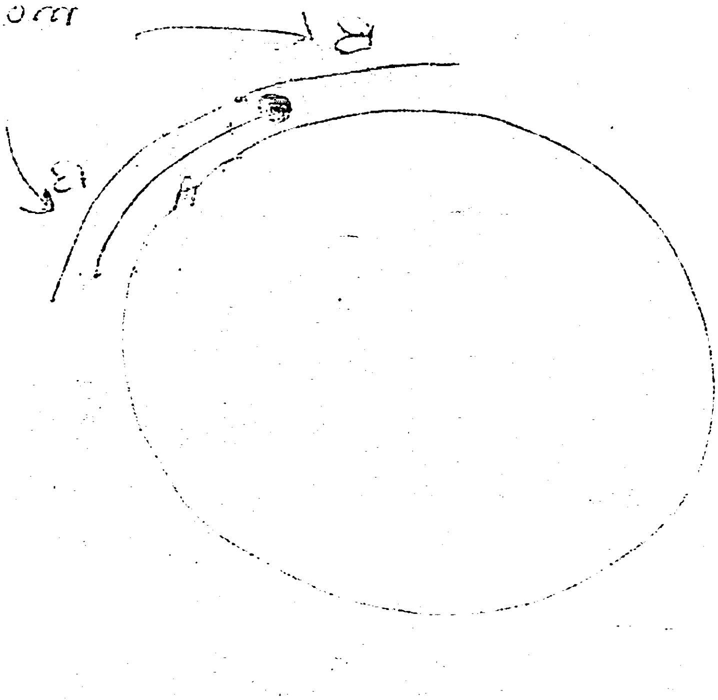
$$k \geq E\Big(\sum_{A \in F} \theta(\pi, A)\Big) = \sum_{A \in F} \frac{k}{\binom{n-1}{k-1}}$$

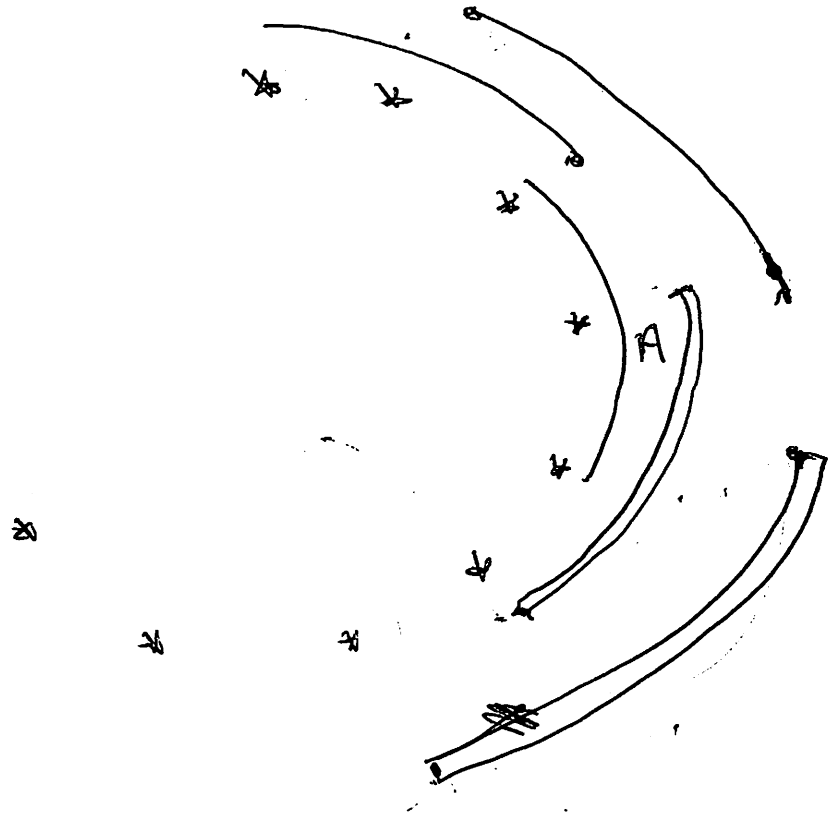
$$= \frac{k}{\binom{n-1}{k-1}} \times |F|$$

$$\Longrightarrow |F| \leq \binom{n-1}{k-1}$$

Claim

$B \in F$ can be in at most one





$k-1 + 1$ possible intervals.

Kraft's Inequality

$X_1, X_2, \ldots, X_m$ is a collection of sequences over an alphabet Σ of size S.

X_i has length n_i. $[n = \max\{n_1, \ldots, n_m\}]$

Assume no X_i is a prefix of another string

$X_i = a_1 a_2 \cdots a_k$

$X_j = a_1 a_2 \cdots a_k a_{k+1} \cdots a_l$

$$\sum_{i=1}^{m} \frac{1}{S^{n_i}} \leq 1$$

Let $\xi = \xi_1 \xi_2 \xi_3 \cdots \xi_n$ be a random string of length n.

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$$\sum_{i=1}^{m} \frac{1}{s^{n_i}} \leq 1$$

Let $\xi = \xi_1 \xi_2 \xi_3 \ldots \xi_n$ be a random string of length n.

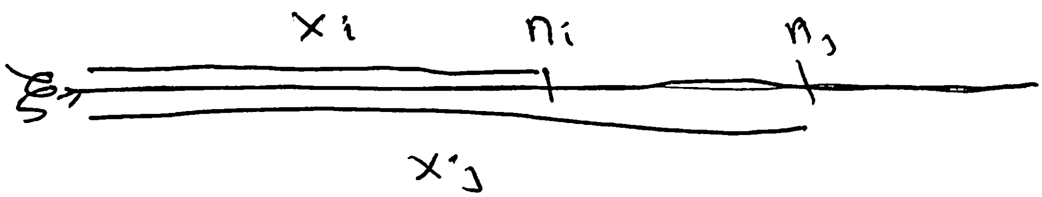
$\mathcal{E}_i = \{ x_i \text{ is the beginning of } \xi \}$

$$\Pr(\mathcal{E}_i) = \frac{1}{s^{n_i}}$$

$\mathcal{E}_i$ and $\mathcal{E}_j$ are disjoint.

$n_i \leq n_j$

Suppose $\mathcal{E}_i \wedge \mathcal{E}_j$ occur.



$\Rightarrow$ x_i is a prefix of x_j — contradiction

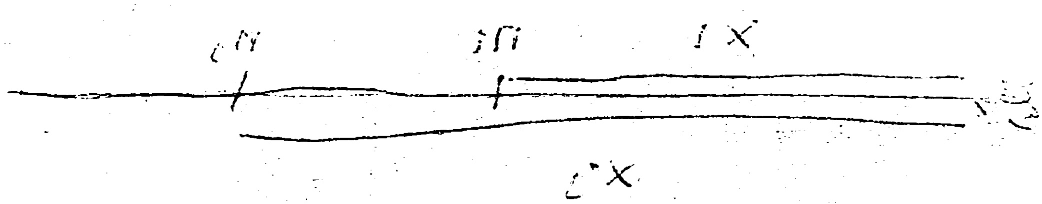
$B_i = \{x_i \text{ is the beginning of } \Sigma\}$

$$P(B_i) = \frac{1}{2^{n_i}}$$

B_i and B_j are disjoint

$n_i \leq n_j$

Suppose $B_i \wedge B_j$ occur



$\Rightarrow$ x_i is a prefix of x_j — contradiction

X is a set and $\mathcal{F} \subseteq 2^X$

For $Y \subseteq X$ we let $\mathcal{F} \cap Y = \{F \cap Y : F \in \mathcal{F}\}$.

For positive integer k

$$f_{\mathcal{F}}(k) = \max\{|\mathcal{F} \cap Y| : |Y| = k\}$$
$$\le 2^k$$

$$t_r(\mathcal{F}) = \max\{m : f_{\mathcal{F}}(m) = 2^m\}$$

[if $|\mathcal{F} \cap Y| = 2^k$, we say $\mathcal{F}$ <u>shatters</u> Y]

$X = \mathbb{R}^2$ - 2 dimensional Euclidean space

$\mathcal{F} = \{ \tfrac{1}{2}\text{-spaces} \}$

$S =$  ✓

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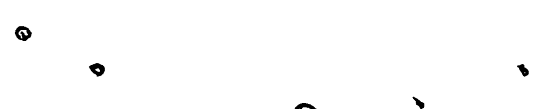
$S =$ 

$S =$  X

IV

X is a set and $\mathcal{F} \subseteq 2^X$

For $Y \subseteq X$ we let $\mathcal{F}_{|Y} = \{F \cap Y : F \in \mathcal{F}\}$

For positive integer k

$$\pi_{\mathcal{F}}(k) = \max\{|\mathcal{F} \cap Y| : |Y| = k\}$$

$$\leq 2^k$$

$$d(\mathcal{F}) = \max\{m : \pi_{\mathcal{F}}(m) = 2^m\}$$

[$|\mathcal{F} \cap Y| = 2^k$, we say $\mathcal{F}$ <u>shatters</u> Y]

$X = \mathbb{R}^2$ - 2 dimensional Euclidean space

$\mathcal{H} = \{\tfrac{1}{2}\text{-space}\}$