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Linear Algebraic Methods

Odd town

Citizenry = $[n]$

There are clubs $C_1, C_2, \ldots, C_m \subseteq [n]$

Satisfying

(a) Each club has an odd number of members

(b) $|C_i \cap C_j|$ is even for $i \neq j$

Claim: $m \leq n$.

Let

$$\underline{V}_i = (V_{i,1}, V_{i,2}, \ldots, V_{i,n})$$

$=$ incidence vector of club C_i

$C_1 = \{3, 4, 7\}$ & $n = 8$

$\underline{V}_1 = [0, 0, 1, 1, 0, 0, 1, 0]$

Under the rules $V_1, V_2, \ldots, V_n$ are linearly independent over $\mathbb{F}_2 = \{0, 1\}$

Rules: (i) $V_i \cdot V_i = 1, \forall i$, (ii) $V_i \cdot V_j = 0$ $i \neq j$

Suppose that $V_1, V_2, \ldots, V_n$ are linearly dependent over $\mathbb{F}_2$

$$c_1 v_1 + c_2 v_2 + \cdots + c_m v_m = 0$$

Take inner product with v_i

$$c_1 v_1 \cdot v_i + c_2 v_2 \cdot v_i + \cdots + c_m v_m \cdot v_i = 0$$

or

$$c_i = 0$$

So $m \leq n$.

$$A = \{ e_i + e_j : 1 \le i < j \le n \}$$

$$e_i = [0, 0, \ldots, 0, 1, 0, \ldots 0]$$

$$|A| = \frac{n(n-1)}{2} \qquad d_1 = 2 \quad d_2 = \sqrt{2}$$

but

$A \subseteq \{ \sum_{i=1}^{n} x_i = 2 \}$ which has dimension $n-1$.

So ~~A~~ we can transform A to $A' \in \mathbb{R}^{n-1}$ without changing distance.

Point Sets in $\mathbb{R}^n$ with only two distances

$A = \{a_1, a_2, \dots a_m\} \subseteq \mathbb{R}^n$

Suppose first that
$\|a_i - a_j\|$ is the same for all i, j.

Answer is a simplex ie $n+1$.

Suppose now that
$$\|a_i - a_j\| = d_1 \text{ or } d_2$$
~~Let~~ $m(n)$ = maximum size of A.

Theorem

$$\frac{n(n+1)}{2} \leq m(n) \leq \frac{(n+1)(n+4)}{2}$$

Define a multivariate polynomial

$$F(x,y) = (\|x-y\|^2 - d_1^2)(\|x-y\|^2 - d_2^2)$$

$$F(a_i, a_j) = \begin{cases} (d_1 d_2)^2 & i = j \\ 0 & i \neq j \end{cases}$$

Next let

$$f_i(x) = F(x, a_i) \qquad i = 1, 2, \dots, m$$

We will first show that $f_1, f_2, \dots f_m$ are linearly independent over $\mathbb{R}$.

Suppose $\lambda_1 f_1(x) + \lambda_2 f_2(x) + \dots + \lambda_m f_m(x) = 0, \forall x$

Plug in $x = a_j$ and use $f_i(a_j) = \begin{cases} (d_1 d_2)^2 & i = j \\ 0 & i \neq j \end{cases}$

$\Rightarrow$ $$\lambda_i (d_1 d_2)^2 = 0 \qquad \forall i$$

$$\Rightarrow \lambda_i = 0$$

On the other hand each f_i is a linear combination of

$$\left(\sum_{k=1}^{n} x_k^2\right)^2, \left(\sum_{k=1}^{n} x_k^2\right)x_j, x_i, x_i x_j, 1$$

# of these is $1 + n + \frac{n(n+1)}{2} + n + 1$

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