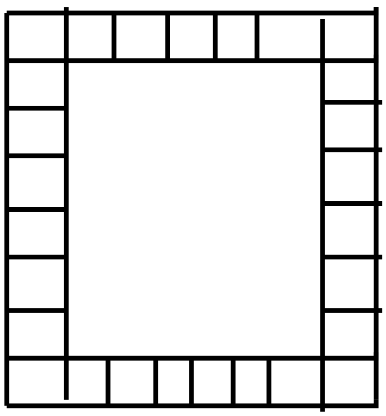
21-301 Combinatorics
Homework 9
Due: Monday, December 3



1. How many ways are there of k-coloring the squares of the above diagram if the group acting is e_0, e_1, e_2, e_3 where e_j is rotation by $2\pi j/4$. Assume that instead of 28 squares there are $4n-4$.

   Solution:

g	e_0	e_1	e_2	e_3
$\lvert Fix(g)\rvert$	k^{4n-4}	k^{n-1}	k^{2n-2}	k^{n-1}

   So the total number of colorings is
   $$\frac{k^{4n-4}+k^{n-1}+k^{2n-2}+k^{n-1}}{4}.$$

2. How many ways are there of k-coloring the squares of the same diagram if the group acting is $e_0, e_1, e_2, e_3, p, q, r, s$ where p, q, r, s are horizontal, vertical, diagonal reflections.

   Solution:

   n **even:**

g	e_0	e_1	e_2	e_3	p	q	r	s
$\lvert Fix(g)\rvert$	k^{4n-4}	k^{n-1}	k^{2n-2}	k^{n-1}	k^{2n-2}	k^{2n-2}	k^{2n-1}	k^{2n-1}

   So the total number of colorings is
   $$\frac{k^{4n-4}+k^{n-1}+k^{2n-2}+k^{n-1}+k^{2n-2}+k^{2n-2}+k^{2n-1}+k^{2n-1}}{8}.$$

   n **odd**

g	e_0	e_1	e_2	e_3	p	q	r	s
$\lvert Fix(g)\rvert$	k^{4n-4}	k^{n-1}	k^{2n-2}	k^{n-1}	k^{2n-1}	k^{2n-1}	k^{2n-1}	k^{2n-1}

   So the total number of colorings is
   $$\frac{k^{4n-4}+k^{n-1}+k^{2n-2}+k^{n-1}+k^{2n-1}+k^{2n-1}+k^{2n-1}+k^{2n-1}}{8}.$$

3. How many ways are there to arrange 2 M's, 4 A's, 5 T's and 6 H's under the condition that any arrangement and its reversal are to be considered the same.

   Solution: The group G consists of $\{e, a\}$ where a is a reflection through the middle of the word. Now

$$\begin{aligned} |Fix(e)| &= \frac{17!}{2!4!5!6!} = 85765680 \\ |Fix(a)| &= \frac{8!}{1!2!2!3!} = 1680 \end{aligned}$$

   A sequence is in $Fix(a)$ if it is a palindrome i.e. looks the same backwards as forwards. It must have middle letter T. Then we arrange 1 M, 2 A's, 2 T"s and 3 H's in any order and then complete the sequence uniquely to a palindrome.

   Thus by Burnside's theorem, the number of sequences is $\frac{85765680+1680}{2} = 42883680$.