21-301 Combinatorics
Homework 8
Due: Monday, November 19

1. In a take-away game, the set S of the possible numbers of chips to remove is finite. Show that the Grundy numbers g satisfy $g(n) \leq |S|$ where n is the number of chips remaining.

   Solution: Observe that for any finite set A, $mex(A) \leq |A|$ since $mex(A) > |A|$ implies that $A \subseteq \{0, 1, 2, \ldots, |A|\}$ which is obviously impossible. In the take-away game $g(n)$ is the mex of a set of size at most $|S|$ and the result follows.

2. Consider the following take-away game: In the first move you are not allowed to take the whole pile. After that, if a player removes x chips, then the next player can remove up to $\lfloor 5x/4 \rfloor$ chips. Determine the P positions.

   Solution: The P-positions, $\{H_1, H_2, \ldots, \}$ satisfy the recurrence

   $$H_{j+1} = H_j + H_k \text{ where } k = \min_{0 \leq \ell \leq j} \{\ell : H_j \leq \lfloor 5H_\ell/4 \rfloor\}. \tag{1}$$

   The first 8 values are given by

j	0	1	2	3	4	5	6	7
H_j	1	2	4	8	16	32	64	128

   We can see that $H_j = 2^j$, but we must prove this by induction. But this follows from

   $$\lfloor 5 \times 2^{j-1}/4 \rfloor < 2^j$$

   which implies that $k = j$ in (1).

3. Find the set of P-positions for the take-away games with subtraction sets

   (a) $S = \{1, 3, 7\}$.

   (b) $S = \{1, 4, 6\}$.

   Suppose now that there are two piles and the rules for each pile are as above. Now find the P positions for the two pile game.

   Solution:

   (a) The first few numbers are

j	0	1	2	3	4	5	6	7	8	9	10
$g_1(j)$	0	1	0	1	0	1	0	1	0	1	0

   It is apparent that $g_1(j) = j \mod 2$ and this follows by an easy induction: If j is even then $j - x, x \in S$ is odd and if j is odd then $j - x, x \in S$ is even.

   (b) The first few numbers are

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$g_2(j)$	0	1	0	1	2	0	1	0	1	2	0	1	0	1	2

So, we see that the pattern 0 1 0 1 2 repeats itself. Again, induction can be used to verify that this continues indefinitely.

(c) The P-positions are those j, k for which $g_1(j) \oplus g_2(k) = 0$. Thus

$$P = \{(j,k) : (k \mod 10 \leq 3) \text{ and } (j \mod 2 = k \mod 10)\}.$$