21-301 Combinatorics
Homework 7
Due: Monday, November 12

1. Let $r_n = r(3, 3, \ldots, 3)$ be the minimum integer such that if we n-color the edges of the complete graph K_N there is a monochromatic triangle.

   (a) Show that $r_n \leq n(r_{n-1} - 1) + 2$.

   (b) Using $r_2 = 6$, show that $r_n \leq \lfloor n!e \rfloor + 1$.

   Solution: Let $N = n(r_{n-1} - 1) + 2$ and consider an n-coloring σ of the edges of K_N. Now consider the $N - 1$ edges incident to vertex N. There must be a color, n say, that is used at least r_{n-1} times, Pigeon Hole Principle. Now let $V \subseteq [N - 1]$ denote the set of vertices v for which the edge $\{v, N\}$ is colored n. Consider the coloring of the edges of V induced by σ. If one of these $\{v_1, v_2\}$ has color N then it makes a triangle v_1, v_2, N with 3 edges colored n. Otherwise the edges of V only use $n - 1$ colors and since $|V| \geq r_{n-1}$ we see by induction that V contains a mono-chromatic triangle.

(b) Using $r_2 = 6$, show that $r_n \leq \lfloor n!e \rfloor + 1$.

Solution: Divide the inequality (a) by $n!$ and putting $s_n = r_n/n!$ we obtain

$$s_n \leq s_{n-1} - \frac{1}{(n-1)!} + \frac{2}{n!}. \tag{1}$$

We write this as

$$\begin{aligned} s_n - s_{n-1} &\leq -\frac{1}{(n-1)!} + \frac{2}{n!} \\ s_{n-1} - s_{n-2} &\leq -\frac{1}{(n-2)!} + \frac{2}{(n-1)!} \\ &\vdots \\ s_3 - s_2 &\leq -\frac{1}{1!} + \frac{2}{2!} \end{aligned}$$

Summing gives

$$s_n - s_2 \leq -1 + \frac{1}{n!} + \sum_{k=2}^{n} \frac{1}{k!} \leq -1 + \frac{1}{n!} + e - 2.$$

Now $s_2 = 3$ and multiplying the above by $n!$ gives $r_n \leq n!e + 1$. We round down, as r_n is an integer.

2. Show that if the edges of K_{m+n} are colored red and blue then either (i) there is a red path with m edges or (ii) a vertex of blue degree at least n.

   Solution: If there is no vertex of blue degree at least n then the red graph has minimum degree at least m. Let $P = x_1, x_2, \ldots, x_k$ be a longest path in the red graph. All of x_k's neighbors in the red graph lie on P, else P can be extended. But x_k has at least m neighbours and so $k \geq m + 1$.

3. Given a set I of n intervals in R, assume that there is no nested set of intervals with size k (a set of intervals are nested if for every pair, one is completely contained inside the other). Then prove that there exists a subset of size $\lceil n/k \rceil$ where no pair of intervals are nested.

   Solution: The nesting property defines a partial order. By Dilworths theorem, if the longest chain has size k, the set of intervals can be partitioned into k sets where each set is an anti-chain. One such anti-chain has size at least $\lceil n/k \rceil$.