21-301 Combinatorics
Homework 6
Due: Monday, November 5

1. Let G be a bipartite graph with bipartition A, B where $|A| = |B| = n$. Let m be the number of edges of G. Show that if $r \geq 2$ and
$$n\binom{m/n}{2} > (r-1)\binom{n}{2} \tag{1}$$
then G contains a copy of $K_{2,r}$. Here $K_{2,r}$ is a bipartite graph with vertex set X, Y where $|X| = 2, |Y| = r$ and edge set $X \times Y$.

   Solution: A vertex $b \in B$ is the common neighbor of $\binom{d(b)}{2}$ pairs of vertices in A, where $d(.)$ denotes degree. If
$$\sum_{b\in B}\binom{d(b)}{2} > r-1)\binom{n}{2} \tag{2}$$
then there is a pair of vertices $a_1, a_2 \in A$ that are the common neighbor of r vertices in B. This implies that G contains a copy of $K_{2,r}$. The LHS of (1) is a lower bound on the LHS of (2).

2. Use the pigeon-hole principle to show that for every integer $k \geq 1$ and prime $p \neq 2, 5$ there exists a power of p that ends with $000\cdots0001$ (k 0's).

   Solution: If we consider the infinite sequence $u_\ell = p^\ell \mod 10^{k+1}$ for $\ell = 1, 2, \ldots,$ then by the PHP there exist $m < n$ such that $u_m = u_n$, In which case,
$$p^n - p^m = 10^{k+1}s \text{ or } p^{n-m}(p^m - 1) = 10^{k+1}s$$
for some positive integer s.

   Now p and 10 are co-prime and therefore $p^m - 1 = 10^{k+1}s'$ for some positive integer s', and this implies the result.

3. Suppose we 2-color the edges of K_6 Red and Blue. Show that there are at least two monochromatic triangles.

   Solution: Assume w.l.o.g. that triangle $(1, 2, 3)$ is Red and that $(4, 5, 6)$ is not Red and in particular that edge $(4, 5)$ is Blue. If $x = 4, 5$ or 6 then there can be at most one Red edge joining x to $1, 2, 3$, else we get a Red triangle. So we can assume that there are two Blue edges joining each of $4, 5$ to $1, 2, 3$. So there must be $x \in \{1, 2, 3\}$ such that both $(x, 4)$ and $(x, 5)$ are Blue. But then triangle $(x, 4, 5)$ is Blue.