21-301 Combinatorics
Homework 5
Due: Monday, October 29

1. Let $S_1, S_2, \ldots, S_m$ and $T_1, T_2, \ldots, T_m$ be two partitions of the set X into sets of size k. Show that there is a set $\{s_1, s_2, \ldots, s_m\}$ that is a set of distinct representatives for both $S_1, S_2, \ldots, S_m$ and $T_1, T_2, \ldots, T_m$.

   Solution Consider the bipartite (multi-)graph G with vertex set $A = B = [m]$ and an edge (i, j) for each $x \in S_i \cap T_j$. This graph is k-regular and so has a perfect matching which we will write as $e_i = (i, \pi(i))$ for $i \in [m]$. Now choose an element $x_i \in S_i \cap T_{\pi(i)}$ for each edge e_i. Clearly $x_i \in S_i$ and $x_i \in T_{\pi(i)}$ for $i \in [m]$. So the only thing we need to do is to show that the x_i are distinct. But if $x_i = x_j$ then $S_i \cap S_j \neq \emptyset$ and then $S_1, S_2, \ldots, S_m$ is not a partition, contradiction.

2. Show that a tree has at most one perfect matching.

   Solution If a tree has two distinct perfect matchings M_1, M_2 then there symmetric difference $M_1 \oplus M_2$ consists of disjoint (alternating) cycles. But a tree has no cycles.

3. Let G be a bipartite graph with bipartition X, Y such that the degree $d(x) \geq 1$ for all $x \in X$ and $d(x) \geq d(y)$ for all edges (x, y) of G. Show that G has a matching that covers every vertex of X.
   (Hint: Suppose there is no such matching. Consider $S \subseteq X$ with fewer than $|S|$ neighbours and as small as possible.)

   Solution Suppose that there is no matching covering X. Then by Hall's Theorem, there exists a *witness* $S \subseteq X$ such that $|S| > |N(S)|$. Assume that S is as small as possible and the $T = N(S)$. Then we have $|S| = |T| + 1$, else we can delete an element of S and find a smaller witness. Now let $S' = S \setminus \{s\}$ for some $s \in S$. Then we have $|A| \leq |N(A)|$ for all $A \subseteq S'$ else A will be a smaller witness than S. So by Hall's theorem, there is a perfect matching of S' into T. But this implies that

   $$\sum_{x \in S} d(x) = d(s) + \sum_{x \in S'} d(x) > \sum_{x \in S'} d(x) \geq \sum_{y \in T} d(y).$$

   But if $E(S : T)$ denotes the set of edges from S to T then

   $$\sum_{x \in S} d(x) = |E(S : T)| \leq \sum_{y \in T} d(y),$$

   contradiction.