Homework 4

Due Wednesday, October 10th,

1. In the kingdom of Far Far Away there are coins of values 1, 2 and 3 dollars. In how many ways can the people of Far Far Away change n dollars?

   Hint:
$$\frac{1}{1+x+x^2} =$$
$$\sum_{n=0}^{\infty}\left(\left(\frac{1}{2}+\frac{\sqrt{3}i}{6}\right)\left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)^n+\left(\frac{1}{2}-\frac{\sqrt{3}i}{6}\right)\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^n\right)x^n.$$

   Answer: The number of ways is the same as the number of solutions for $x_1+x_2+x_3=n$ where $x_i \equiv 0 \mod i$. Using the scheme discussed in class we may write the generating function
$$a(x) = (1+x+x^2+x^3+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots) =$$
$$= \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3} = \frac{1}{1-x}\cdot\frac{1}{(1-x)(1+x)}\cdot\frac{1}{(1-x)(1+x+x^2)} =$$
$$= \frac{1}{(1-x)^3}\cdot\frac{1}{1+x}\cdot\frac{1}{1+x+x^2} = \frac{Ax^2+Bx+C}{(1-x)^3}+\frac{D}{1+x}+\frac{Ex+F}{1+x+x^2}.$$
   Solving six linear equations with six variables we get that the generating function is
$$a(x) = \frac{17x^2-52x+47}{72(1-x)^3}+\frac{1}{8(1+x)}+\frac{x+2}{9(1+x+x^2)}.$$
   Using the generalized binomial theorem (and the hint) we get that the coefficient of x_n is
$$\frac{17}{72}\binom{n}{2}-\frac{52}{72}\binom{n+1}{2}+\frac{47}{72}\binom{n+2}{2}+\frac{1}{8}(-1)^n+\frac{2}{9}(a\cdot(c^n+c^{n-1})+b\cdot(d^n+d^{n-1}))$$
   where
$$a=\left(\frac{1}{2}+\frac{\sqrt{3}i}{6}\right), c=\left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right), b=\left(\frac{1}{2}-\frac{\sqrt{3}i}{6}\right), d=\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right).$$

2. Assume you are given n *labeled* bills of value 1 dollar, m labeled bills of value 2 dollars and p labeled bills of value 3 dollars. Write the generating function for the number of ways in which you can change N dollars given these labeled bills. You do not have to write the generating function as a power series.

   Answer: Here every bill is considered different, so we take a different variable for every bill. Each of these variables has two values — one when we use the corresponding bill and one when we do not. Hence the generating function for the problem is
$$(1+x)^n(1+x^2)^m(1+x^3)^p$$

3. Let $B(n)$ be the number of ways in which one can put n labeled balls into unlabeled sacks. For instance balls $\{1, 2, 3\}$ could be split in five ways:

$$\{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{1\}, \{2, 3\}\}, \{\{1, 2, 3\}\}.$$

Find a recursion for $B(n)$. You do not need to solve the recursion.

Answer: Let k be the number of balls that are in the same sack as the $n+1$'st ball. There are $\binom{n}{k}$ choices for the balls with $n+1$, and $B(n-k)$ ways to put the other balls in sacks. All in all we get

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(n-k) = \sum_{k=0}^{n} \binom{n}{k} B(k).$$

Finally, there is one empty set, so $B(0) = 1$.