Homework 3 solutions

1. Label the couples $1, 2, ..., n$. Let A_S be the set of all configurations in which the couples in set S sit in front of each other. There are $\binom{n}{|S|}$ possible seats for these couples. Also we have $2^{|S|}$ ways to sit the males and females on the two benches, and another $|S|!$ to choose which couple sits where. There are $(2n - 2|S|)!$ ways to seat the remaining people. So $|A_S| = \binom{n}{|S|} 2^{|S|}(2n - 2|S|)!$ So, by the inclusion exclusion principle we get:

$$\left|\bigcap \overline{A_i}\right| = \sum_{k=0}^{n} \binom{n}{k}^2 2^k k!(2n-2k)!.$$

2. (a) Let a_n be the number of strings made of zeros and ones with no two consecutive ones. If a_n ends in a 0, we have a_{n-1} possible strings. If a_n ends in a 1, it must end in a 01, so we have a_{n-2} possible strings. So,

$$a_n = a_{n-1} + a_{n-2}.$$

   There is one empty velid sequence, two valid sequences of length 1 and three of length 2.

   Therefore $a_n = F_{n+1}$, where F_n is the n'th Fibonacci number.

   (b) Let $\{a_n\}$ be a string of length n that satisfies the condition in the problem. Define $\{b_{n-1}\}$ as follows: $b_i = 1$ iff $a_i = a_{i+1}$ and 0 otherwise. The string $\{b_{n-1}\}$ has no two consecutive ones. From (a) above, there are F_n strings of the defined type. For each string b_{n-1} there are 2 strings a_n. So, the answer is $2F_n$.

3. Let $F(x) = \sum a_n x^n$. Then

$$F(x) - 2 - 10x = 6x(F(x) - 2) + 7x^2 F(x),$$

$$F(x) = \frac{2 - 2x}{1 - 6x - 7x^2},$$

$$F(x) = \frac{3/2}{1 - 7x} + \frac{1/2}{1 + x}.$$

   So

$$a_n = \frac{3}{2}7^n + \frac{1}{2}(-1)^n.$$