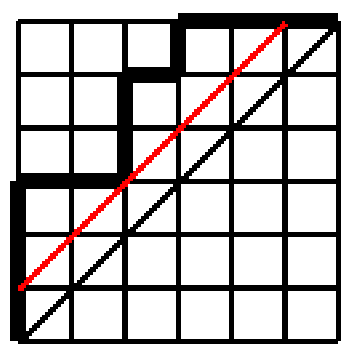
Solutions to homework 2

1.

$$\sum_{0\le i\le k\le n}\binom{n}{k}\binom{k}{i}=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{k}\binom{k}{i}=\sum_{k=0}^{n}2^k\binom{n}{k}=(1+2)^n=3^n$$

2. We define $\mathrm{PATH}_>(n,n)$ as the set of paths in an n by n grid that are strictly above the diagonal. That is, $\mathrm{PATH}_>(n,n)$ is the set of all paths in $\mathrm{PATH}_\ge(n,n)$ that do not intersect the diagonal (except for the endpoints). We observe that $|\mathrm{PATH}_>(n,n)|=|\mathrm{PATH}_\ge(n-1,n-1)|$. This can be seen from the picture. By moving the diagonal one square up, we can transform a path in $\mathrm{PATH}_>(n,n)$ into a path in $\mathrm{PATH}_\ge(n-1,n-1)$ and viceversa. Formally, for a path $P\in\mathrm{PATH}_>(n,n)$ we remove the first step (which is always $(0,0)\to(0,1)$) and the last step (which is always $(n-1,n)\to(n,n)$) to get a path $P'\in\mathrm{PATH}_\ge(n-1,n-1)$.



The number of paths in $\mathrm{PATH}_\ge(n+1,n+1)$ that intersect the diagonal for the first time after the origin in the point (k,k) is

$$|\mathrm{PATH}_>(k,k)|\cdot|\mathrm{PATH}_\ge(n+1-k,n+1-k)|=$$
$$|\mathrm{PATH}_\ge(k-1,k-1)|\cdot|\mathrm{PATH}_\ge(n+1-k,n+1-k)|=C_{k-1}C_{n+1-k}.$$

Since all paths in $\mathrm{PATH}_\ge(n+1,n+1)$ either do not intersect the diagonal or first intersect it in some point (i,i) we have that $C_{n+1}=\sum_{i=0}^{n}C_iC_{n-i}$. The term with $i=0$ corresponds to the number of paths in a $n+1$ by $n+1$ grid that stay strictly above the diagonal.

3. Answer: The number of non-crossing handshakes between $2n$ people is $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Let $S(n)$ be the number of non-crossing handshakes between $2n$ people. We have that $S(1) = 1 = C_1$. Also, we define $S(0) = 1 = C_0$.

Assume we have a configuration of non-crossing handshakes. Suppose person 1 shakes hands with person i. Then, persons $2, \dots i-1$ must form a configuration of non-crossing handshakes. So. i must be even. Let $i = 2k$. Also, persons $i+1, i+2, \dots 2n$ must form a configuration of non-crossing handshakes. So, the number of configurations of non-crossing handshakes with $2n$ in which person i and person $2k$ shake hands is $S(k-1)S(n-k)$. Person 1 must shake hands with somebody, so $S(n) = \sum_{i=1}^{n} S(i-1)S(n-i)$. This is the same recursion as the one proven in problem 2 for Catalan numbers. We also have that $S(0) = C_0, S(1) = C_1$. Therefore, $S(n) = C_n$.