21-301 Combinatorics
Homework 1
Due: Wednesday, September 5

1. How many integral solutions of

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100$$

satisfy $x_1 \geq 6$, $x_2 \geq 10$, $x_3 \geq -3$, $x_4 \geq 4$ and $x_5 \geq 4$?
Solution Let

$$y_1 = x_1 - 6, \qquad y_2 = x_2 - 10, \qquad y_3 = x_3 + 3, \qquad y_4 = x_4 - 4, \qquad y_5 = x_5 - 4.$$

An integral solution of $x_1+x_2+x_3+x_4+x_5 = 100$ such that $x_1 \geq 6$, $x_2 \geq 10$, $x_3 \geq -3$, $x_4 \geq 4$ and $x_5 \geq 4$ corresponds to an integral solution of $y_1 + y_2 + y_3 + y_4 + y_5 = 79$ such that $y_1, \ldots, y_5 \geq 0$. From a result in class,

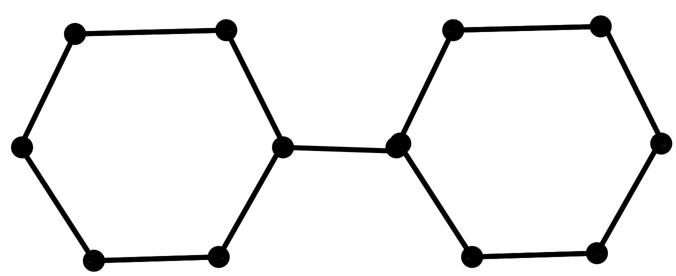
$$|\{(y_1, y_2, y_3, y_4, y_5) : y_1, \ldots, y_5 \in Z_+ \text{ and } y_1 + \cdots + y_5 = 79\}| = \binom{79+5-1}{5-1} = \binom{83}{4}.$$

2. Show that

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell} 2^{n-\ell}.$$

Solution The LHS is the number of choices of pairs (A, B) where $A, B \subseteq [n]$ and $A \subseteq B$ and $|A| = \ell$. The RHS is the number of choices of pairs (A, C) where $A \subseteq [n], |A| = \ell$ and $C \subseteq [n] \setminus A$. The map $(A, B) \to (A, B \setminus A)$ is a bijection that shows the two sides are equal.

3. How many ways are there of placing k 1's and $2n - k$ 0's at the vertices of the cycles in the diagram below so that each 1 is separated by at least one 0?



Each cycle has n vertices and so there are $2n$ vertices altogether.

Solution In class it was shown that the number of ways to place k 1's ($k > 0$) and $n - k$ 0's on the vertices of a polygon such that no two ones are adjacent is $\frac{n}{k}\binom{n-k-1}{k-1}$. Let r be the number of 1's in the left cycle, then the number of ways of placing k 1's and $2n - k$ 0's such that no two ones *on the same cycle* are adjacent is

$$2\frac{n}{k}\binom{n-k-1}{k-1} + \sum_{r=1}^{k-1} \frac{n}{r}\binom{n-r-1}{r-1}\frac{n}{k-r}\binom{n-(k-r)-1}{k-r-1}$$

(We need to take special care for the cases $r = 0$ and $r = k$, which reduce to the case shown in class).

Next we need to count the number of settings in which the two endpoints of the bridge are marked by 1. Again let $1 \leq r \leq k-1$ be the number of ones in the left cycle, and let $a_1, \ldots, a_r$ be the number of zeros between two ones counting clockwise from the one on the bridge. The number of settings is then identical to the number of solutions for $a_1 + \cdots + a_r = n - k$ where $a_i \geq 1$ for all $1 \leq i \leq r$. This number is $\binom{n-k-r+r-1}{r-1} = \binom{n-k-1}{r-1}$. The same reasoning shows that the number of settings in the right cycle is $\binom{n-k-1}{k-r-1}$.

All in all we get

$$2\frac{n}{k}\binom{n-k-1}{k-1} + \sum_{r=1}^{k-1} \frac{n}{r}\binom{n-r-1}{r-1}\frac{n}{k-r}\binom{n-(k-r)-1}{k-r-1} - \binom{n-k-1}{r-1}\binom{n-k-1}{k-r-1}.$$