11\9\12

Grundy Numbers

Game represented by $D=(V,A)$. Grundy number g satisfy

$$g(x) = \text{mex}\{\, g(y) : y \in N^+(A) \}$$

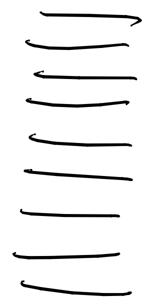
Observe

$$x \in P \longleftrightarrow g(x) = 0 \qquad |V| = n$$

$$g(x) = 0 \longleftrightarrow g(y) > 0, \quad \forall\, y \in N^+(x) \qquad g(n) = 0$$

$$x \in P \longleftrightarrow y \in N, \quad \forall\, y \in N^+(x) \qquad n \in P$$

Grundy Numbers of Simple Game



Rules: (i) a player can remove any even no. of chips, but not the whole pile, if its even.

(ii) A player can remove whole pile if it is odd.

Terminal positions are 0 & 2.

Claim $g(0)=0$

$g(2k)=k-1$

$g(2k-1)=k$

n	0	1	2	3	4	5	6	7	8	9
$g(n)$	0	1	0	2	1	3	2	4	3	5

Proof of claim: by induction on k.

True for small $k \leq 4$

Assume true for $k \leq 4$.

$$g(2k) = \text{mex}\{g(2k-2), g(2k-4), \dots g(2)\}$$

$$= \text{mex}\{k-2, k-3, \dots \dots 0\}$$

$$= k-1$$

$$g(2k-1) = \text{mex}\{g(2k-3), g(2k-5), \dots g(1), g(0)\}$$

$$= \text{mex}\{k-1, k-2, \dots \dots 1, 0\}$$

$$= k.$$

Sum of Games

$a = a_k a_{k-1} \cdots a_0$ binary

$b = b_k b_{k-1} \cdots b_0$

$$a \oplus b = \begin{array}{cccc} a_k & a_{k-1} & \cdots & a_0 \\ \oplus & \oplus & \cdot & \oplus \\ b_k & b_{k-1} & & b_0 \end{array}$$

exclusive or

$x = (x_1, x_2, \dots, x_p)$ is a position in $G_1 \oplus G_2 \oplus \cdots \oplus G_p$

then $g(x) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_p(x_p)$

where g_i is the Grundy function for G_i

Proof

Can assume that $p = 2$ and apply induction.

$p = 3$ $\qquad g(x) = \underbrace{\left(g_1(x_1) \oplus g_2(x_2)\right)}_{(G_1 \oplus G_2)} \overset{\oplus}{\oplus} \overset{G_3}{g_3(x_3)}$

$G = G_1 \oplus G_2$

Show $g(x_1, x_2) = g_1(x_1) \oplus g_2(x_2)$

[Backwards induction on topological order of $D_1 \times D_2$]

Suppose that $g_1(x_1) \oplus g_2(x_2) = b$

We must show that $b = \operatorname{mex}\{ g_1(x_1') \oplus g_2(x_2') : (x_1', x_2') \in N^+(x_1, x_2)$

A1: if $x \in X$ and $g(x) = b > a$ then $\exists\, x' \in N^+(x)$ s.t. $g(x') = a$.

A2: if $x \in X$ and $g(x) = b$ and $x' \in N^+(x)$ then $g(x') \neq b$

A3: if $x \in X$ and $g(x) = 0$ and $x' \in N^+(x)$ then $g(x') \neq 0$

SAME

A1:

$$\underbrace{g_1(x_1) \oplus g_2(x_2) \oplus \underbrace{(a \oplus b)}_{d}}_{\text{if this is } < g(x_2)} = a$$

then $\exists\ x_2' \in N^+(x_2)$ s.t.

$$g_2(x_2') = g_2(x_2) \oplus d$$

$$\Longrightarrow \quad g(x_1, x_2') = a$$

Show either $\quad g_1(x_1) \oplus d < g_1(x_1)$

or $\quad g_2(x_2) \oplus d < g_2(x_2)$

Suppose $2^{k-1} \leq d < 2^k$

$d = 0\,0\,0 \ldots \underset{\substack{\uparrow \\ \underline{1}}}{d_k}\, d_{k-1} \ldots d_1\, d_0$ in binary

$d_k = \underline{1} = a_k \oplus b_k \Rightarrow a_k = 0$ & $b_k = \underline{1}$

$[a_k = \underline{1}$ & $b_k = 0 \Rightarrow a > b]$

$\underline{1} = b_k = g_1(x_1)_k \oplus g_2(x_2)_k$

assume $\underline{1}$ 0

$\rightarrow$ $g_1(x_1) \oplus d < g_1(x_1)$ because d "kills" bit k

A2:
& A_3 $g_1(x_1') \oplus g_2(x_2) = g_1(x_1) \oplus g_2(x_2) = b$

$\Longrightarrow$ $g_1(x_1') = g_1(x_1)$ — contradition

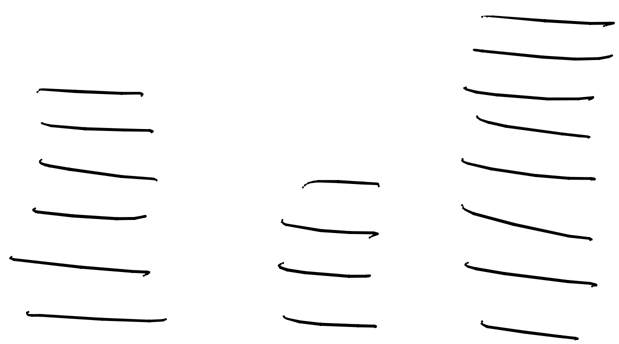
g_1 doesn't "see" same number.

Suppose we have 2 piles & $S = \{1, 2, 3, 4\}$ in both piles

$g_1(n) = n \mod 5 = g_2(n)$ [Check]

$(n_1, n_2) \in P \leftrightarrow n_1 = n_2 \mod 5$

$n_1 \neq n_2 \mod 5$ then $g(n_1, n_2) \neq 0$.

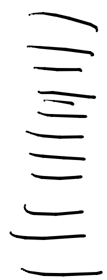


S=[3] S=[2] S=[4]

P-position

0	1	0
0	0	1
0	1	1
0	0	0

More complicated take-away game



First move, can take $1, 2, \ldots, n-1$ chips

Subsequently: if player takes x chips

next player takes $y \le x$ chips

Losing positions are powers of 2.

10

1010

Take lowest order 1 bit

1000

0111 0110

# of 1's does not decrease

General strategy: take lowest bit.

Can't do this for a power of 2.